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535. INEQUALITIES INVOLVING THE AREA OF A QUADRILATERAL INSCRIBED IN A CONVEX QUADRILATERAL*

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1. Let $PQRS$ be inscribed in the convex quadrilateral $ABCD$ with P on AB, Q on BC etc, and let $AP/PB=p$, $BQ/QC=q$ etc. It is assumed no two vertices coincide. The area of a convex polygon with vertices $P_1, P_2, \dots, P_n$ will be denoted $|P_1P_2\cdots P_n|$. Let $|DAB|=A_1$, $|ABC|=A_2$, $|BCD|=A_3$, $|CDA|=A_4$. Then,

$$|SAP|=\frac{p}{p+1}|SAB|=\frac{pA_1}{(1+p)\,(1+s)}.$$

Since $|ABCD|=\frac{1}{2}(A_1+A_2+A_3+A_4)$ it follows,

$$V\equiv|PQRS|=\sum\left(\frac{1}{2}-\frac{p}{(1+p)\,(1+s)}\right)A_1\equiv f_1A_1+f_2A_2+f_3A_3+f_4A_4$$

where the coefficients $f_1, \dots, f_4$ depend only on the ratios p, q, r, s.

Theorem 1. *$V\equiv|PQRS|$ satisfies the following inequalities*:

$$(1)\qquad V\leqq\left[1+\frac{(1-pr)\,(1-qs)}{\prod(1+p)}\right]\max(A_i)\equiv[1+(1-p_1-r_1)\,(1-q_1-s_1)]\max(A_i),$$

$$(2)\qquad V\geqq\left[1+\frac{(1-pr)\,(1-qs)}{\prod(1+p)}\right]\min(A_i)\equiv[1+(1-p_1-r_1)\,(1-q_1-s_1)]\min(A_i)$$

where $p_1=p/(1+p)=AP/AB$ etc.

Proof. First, the coefficients f_i have the property that $f_i+f_j>0$ for $i\in\{1, 3\}$ and $j\in\{2, 4\}$. For since $f_1=\frac{1}{2}-p/(1+p)(1+s)$ and $f_2=\frac{1}{2}-q/(1+q)(1+p)$ we have $f_1+f_2=(1+s+ps+pqs)/(1+p)(1+q)(1+s)>0$ and the other cases follow similarly.

Suppose now that p, q, r, s are fixed, that $\max(A_i)=k$, and that V is maximised for A_i subject only to this condition. We show that then all the A_i must be equal to k. Obviously one must be, say A_4. If $A_2<k$ then since geometrically $A_1+A_3=A_2+A_4$ one of A_1, A_3 is less than k, say $A_1<k$. But $f_1+f_2>0$ so that V could be increased if A_1, A_2 were replaced by A_1+x, A_2+x, for a suitably small $x>0$. So $A_2=k$ which implies $A_1=A_3=k$.

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Thus V is maximised only when $A_1 = A_2 = A_3 = A_4 = k$ so that

$$V = (f_1 + f_2 + f_3 + f_4)\, k.$$

A computation shows $\sum f_i = 1 + (1-pr)(1-qs)/\prod(1+p)$ whence (1) follows. (2) can be proved similarly by minimising V subject to min $(A_i) = k$ and one finds again that $A_1 = \cdots = A_4 = k$. *Q.E.D.*

Corollary (a). *Equality holds in* (1) *or* (2) *if and only if all A_i are equal i. e. if and only if ABCD is a parallelogram.*

Corollary (b). *From* (1) *it follows that if $pqrs = 1$ then $V \leqq \max(A_i)$ with equality if and only if ABCD is a parallelogram and $pr = qs = 1$ and from* (2) *if $pr = qs$ then $V \geqq \min(A_i)$ with equality if and only if ABCD is a parallelogram and $pr = qs = 1$. The condition $pqrs = 1$ can be expressed geometricaly a simple application of Menelaus' theorem shews it is equivalent to SP, RQ meeting on DB (or equally PQ, SR meeting on AC).*

2. The plane section of largest area of a tetrahedron

Several proofs have been published of the following theorem ([**1**], [**2**], [**3**]) but a proof based on the corollary to (1) above appears to be new.

Theorem 2. *The plane section of largest area of a tetrahedron is a face.*

Proof. Let the tetrahedron be $ABCD$ with largest face area f and let W be a plane section, of area $|W|$. If W is triangular and not a face then $|W|$ could be increased by moving a vertex of W along an edge of $ABCD$ until it coincided with a vertex of the latter. Thus W must already be a face.

If W is quadrilateral let it meet AB in P, BC in Q, CD in R, DA in S and let $AP/PB = p \cdots DS/SA = s$. Let $ABCD$ be projected perpendicularly onto W, into $A'B'C'D'$. The inequality

$$\text{(3)} \qquad |W| < \max(|A'B'C'|, |B'C'D'|, |C'D'A'|, |D'A'B'|)$$
$$\leqq \max(|ABC|, \ldots, |DAB|) = f$$

is easily seen to hold in all cases of the resulting configuration except possibly in the case when $A'B'C'D'$ (in that order) forms a convex quadrilateral. In that case we have by similar triangles that $p = AP/PB = A'P/PB' = AA'/BB'$ so that $pqrs = 1$ and then (3) follows when $|A'B'C'|, \ldots, |D'A'B'|$ are not all equal by applying corollary (b) to theorem 1 to the quadrilateral $PQRS$ inscribed in $A'B'C'D'$. Since one of the faces of $ABCD$ must have larger area than its projection it follows $|W| < f$ in all cases. *Q.E.D.*

The analogue of theorem 2 is true in 4 dimensions ([**3**]) although false for higher dimensions ([**3**], [**4**]) and analogously to the inequality used in the proof above there is an inequality in 3-dimensions for a triangular prism inscribed in a triangular faced hexahedron. This configuration can arise when one projects a 4-simplex into a solid section of it. More precisely let $XYZTU$ be the hexahedron, with faces TXY, TYZ, TZX, UXY, UYZ, UZX and let $ABCDEF$ be the prism with end faces ABC, DEF where A, B, C, D, E, F lie respectivly on UX, UY, UZ, TX, TY, TZ. Let $UA/AX=a$, $UB/BY=b$, $UC/CZ=c$, $XD/DT=d$, $YE/ET=e$, $ZF/FT=f$ and suppose that $ad=be=cf$ (as must in general be so if $EDAB$ etc are to be complanar). Let $V=|ABCDEF|$, $V_1=|UYZT|$, $V_2=|UZXT|$, $V_3=|UXYT|$, $V_4=|UXYZ|$, $V_5=|TXYZ|$. Then $V<\max(V_i)$. This can be proved by the same kind of method as was used in proving theorem 1.

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