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534. A PROBLEM ABOUT TAYLOR'S THEOREM*

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1. Let a real valued C^∞ function f be defined in the interval $[0, x]$ so that by TAYLOR's theorem for every $n=1, 2, \ldots$

$$\text{(T)} \qquad f(x)=\sum_{k=0}^{n-1} f^{(k)}(0)\, x^k/k! + f^{(n)}(t_n)\, x^n/n!, \quad 0<t_n<x.$$

When is it possible to choose the sequence $\{t_n\}$ so that $\liminf\ t_n=0$ (this was posed as an unsolved problem in the Amer. Math. Monthly **81** (1974), 1121 by J. A. EIDSWICK). In general this is not possible as the example $f(x)=0,\ x\leqq 1$; $f(x)=\exp\left(-(x-1)^{-2}\right),\ x>1$, with $x=2$ shows but in two cases of interest it can be done.

These are when f is a convergent power series and when f and all its derivatives vanish at 0 but f is not identically zero in any interval $[0, a],\ a>0$.

2. Theorem 1. *Let f be a power series with radius of convergence (r.c.) $\geqq R>x$. Then there is a monotonic increasing function $b(x)$, independent of R, defined on $(0, 1)$ such that t_n can be chosen with*

$$0<t_n<R\, b(x/R)/(n+1)$$

for infinitely many n.

Proof. If f is a polynomial the theorem is trivial. Otherwise, suppose first that $f(x)=\sum c_n x^n$ where $c_n \to 0$ and $0<x<1$. Substituting in (T)

$$\text{(A)} \qquad c_{n+1}x+c_{n+2}x^2+\cdots\cdot=(n+1)\,c_{n+1}\,t_n+\binom{n+2}{2}c_{n+2}\,t_n^2+\cdots$$

Since $c_n \to 0$ there are infinitely many n such that $|c_{n+1}|\geqq|c_{n+k}|$ all $k\geqq 1$, and assuming this (A) can be written,

$$g(x)\equiv x+d_2x^2+d_3x^3+\cdots$$

$$\text{(B)} \qquad =y+d_2(n+1)^{-2}\binom{n+2}{2}y^2+\cdots\equiv F(n, d_i, y)$$

where $t_n=y/(n+1)$ and $d_k=c_{n+k}/c_{n+1}$ with $|d_k|\leqq 1$, all $k\geqq 2$.

Lemma. *Let $F(y)=y+d_2y^2/2!+d_3y^3/3!+\cdots$ where d_i are arbitrary but satisfy $|d_i|\leqq 1$ for all i. Then there exist functions $\delta(x)>0$ and $b_1(x)>0$ not depending on the d_i such that the equations*

$$F(y)=g(x)+\delta(x) \text{ and } F(y)=g(x)-\delta(x)$$

have solutions for y in $(0, b_1(x))$.

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Proof. The proof depends on $g(x)$ being the LAPLACE transform $p\int_0^\infty e^{-pt}F(t)\,dt$ where $p=1/x$. For all $y\geqq 0$, $2y+1-e^y\leqq F(y)\leqq e^y-1$ so that

$$0.09\simeq\alpha=2\,(1/10)+1-e^{1/10}\leqq F(1/10)\leqq e^{1/10}-1\simeq 0.11$$

and $F(\log 2)\geqq 2\log 2+1-2\simeq 0.39$. Also $F(0)=0$.

Choose $\beta>0$ so that $F(y)\leqq e^y-1<\frac{1}{2}\alpha$ in $[0,\beta]$ and set

$$J_1=\int_0^\beta pe^{-pt}\,\frac{1}{2}\,\alpha\,dt;\quad J_2=\int_0^\beta pe^{-pt}\,(2t+1-e^t)\,dt$$

where $p=1/x$.

Put $\delta(x)=\min\left(\frac{1}{2}J_1,\ \frac{1}{2}J_2,\ 1/20\right)$. There are now three cases:

Case 1: $\delta(x)<g(x)\leqq\alpha$. Then $F(y)=g(x)-\delta(x)$ has a solution in $(0, 1/10)$ and $F(y)=g(x)+\delta(x)$ in $(0, \log 2)$.

Case 2: $g(x)>\alpha$. Suppose $F(y)<g(x)+\delta(x)$ in $(0, k)$. Then,

$$\begin{aligned}
g(x)&=\int_0^\infty pe^{-pt}\,F(t)\,dt\\
&\leqq\int_0^\beta pe^{-pt}\,F(t)\,dt+\int_\beta^k pe^{-pt}\,\big(g(x)+\delta(x)\big)\,dt+\int_k^\infty pe^{-pt}\,(e^t-1)\,dt\\
&\leqq\int_0^\beta pe^{-pt}\,\frac{1}{2}\,\alpha\,dt-\int_0^\beta pe^{-pt}\,\big(g(x)+\delta(x)\big)\,dt+\int_k^\infty pe^{-pt}\,(e^t-1)\,dt\\
&\quad+\int_0^\infty pe^{-pt}\,\big(g(x)+\delta(x)\big)\,dt-\int_0^\beta pe^{-pt}\left(g(x)+\delta(x)-\frac{1}{2}\,\alpha\right)dt\\
&\leqq g(x)+\delta(x)-\int_0^\beta pe^{-pt}\,\frac{1}{2}\,\alpha\,dt+\int_k^\infty pe^{-pt}\,(e^t-1)\,dt\\
&\leqq g(x)-\frac{1}{2}J_1+\int_k^\infty pe^{-pt}\,(e^t-1)\,dt
\end{aligned}$$

which is contradictory if $k\geqq k_0(x)$ where $k_0(x)$ does not depend on the d_i. So $F(y)=g(x)+\delta(x)$ must have a solution in $\big(0, k_0(x)\big)$ and since $g(x)-\delta(x)>0$ so also must $F(y)=g(x)-\delta(x)$.

Case 3: $g(x)\leqq\delta(x)$. Suppose that $F(y)>g(x)-\delta(x)$ in $(0, k)$. Then,

$$g(x)=\int_0^\infty pe^{-pt}\,F(t)\,dt$$

$$\geqq \int_0^{\beta} pe^{-pt}(2t+1-e^t)\,dt + \int_{\beta}^{k} pe^{-pt}\big(g(x)-\delta(x)\big)\,dt + \int_k^{\infty} pe^{-pt}(2t+1-e^t)\,dt$$

$$\geqq J_2 + \int_0^{\infty} pe^{-pt}\big(g(x)-\delta(x)\big)\,dt + \int_k^{\infty} pe^{-pt}(2t+1-e^t)\,dt$$

$$\geqq g(x) + \frac{1}{2}J_2 + \int_k^{\infty} pe^{-pt}(2t+1-e^t)\,dt$$

which is contradictory if $k \geqq k_1(x)$ where $k_1(x)$ does not depend on the d_i. So $F(y)=g(x)-\delta(x)$ must have a solution in $(0,\ k_1(x))$ and evidently so must $F(y)=g(x)+\delta(x)$ if $g(x)+\delta(x)\leqq 0$. Otherwise $0<g(x)+\delta(x)\leqq 2\delta(x)\leqq 1/10$ so this equation has a solution in $(0, \log 2)$.

Thus it suffices to take $\delta(x)$ as above and $b_1(x)=\max\big(\log 2,\ k_0(x),\ k_1(x)\big)$ where $b_1(x)$ can obviously be chosen monotonic increasing in $(0, 1)$. This concludes the proof of the lemma.

Next, as $n\to\infty$ (regarding d_i as independent of n), $F(n, d_i, y)\to F(y)$ uniformly in any interval $[0, b]$ and uniformly with respect to the d_i subject to $|d_i|\leqq 1$. So if $F(y_1)=g(x)+\delta(x)$ and $F(y_2)=g(x)-\delta(x)$ where $y_1, y_2 \in \big(0, b_1(x)\big)$ then for $n \geqq n_0(x)$ independent of d_i $F(n, d_i, y)=g(x)$ has a solution y between y_1 and y_2 and so in $\big(0, b_1(x)\big)$. Thus for infinitely many n we may take $0<t_n<b_1(x)/(n+1)$.

If now r.c. $\geqq 1$ and $0<x<1$ but not necessarily $c_n\to 0$ then a change of scale shows that if $\lambda>1$, $\lambda x<1$ then $0<t_n<\lambda^{-1}b_1(\lambda x)/(n+1)$ for infinitely many n. Taking $\lambda = 1+\frac{1}{2}(1-x)$ and putting $b(x)=b_1(\lambda x)$ implies that if r.c. $\geqq 1$ Δy between and $0<x<1$ then $0<t_n<b(x)/(n+1)$ for infinitely many n where $b(x)$ is monotonic increasing in $(0, 1)$. Another scale change then establishes the theorem for r.c. $\geqq R$ and $0<x<R$.

Theorem 2. *Let $f(0)=f'(0)=\cdots=f^{(k)}(0)=0$ for all k. Then either $\{t_n\}$ can be chosen so that $t_n\to 0$ or else $f(x)\equiv 0$ in $[0, a]$ for some $a>0$.*

Proof. Assume $f(x)>0$ (the cases $f(x)=0$ and $f(x)<0$ can be dealt with by similar arguments). Then $f(x)=f^{(n)}(t_n)x^n/n!$ and if for all choices of t_n, $\limsup t_n \geqq 2a>0$ we have $f^{(n)}(t)<n!\,f(x)x^{-n}$ for $0\leqq t\leqq a$ and infinitely many n. Repeated integration then yields $f(t)<f(x)x^{-n}t^n$ for infinitely many n so that if $0\leqq t\leqq a<x$ then $f(t)\leqq 0$. Similarly one finds $f'(t)<nf(x)x^{-n}t^{n-1}$ for infinitely many n whence in $[0, a]$ $f'(t)\leqq 0$. Continuing we deduce $f^{(k)}(t)\leqq 0$ for all $k\geqq 0$ and t in $[0, a]$ so that by BERNSTEIN's theorem $f(x)\equiv 0$ in $[0, a]$.

3. In conclusion we ask whether there are any other simple classes of functions for which similar theorems can be proved, and in particular raise the question of what happens when x is an end point of the interval of convergence of a power series, suitable conditions for the applicability of TAYLOR's theorem being assumed to hold.

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