UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 498—№ 541 (1975), 179—180.

532. REMARK ON AN ELEMENTARY INEQUALITY*

Branko M. Milisavljević

The author of this paper is an undergraduate student at the Technical Faculty University of Novi Sad. This is his first contribution to Mathematics.

Editorial Committee

Let f be a convex and differentiable function on $\mathbf{I}=(a, b)$. Then

$$f(x)+hf'(x)<f(x+h)<f(x)+hf'(x+h) \qquad (h\neq 0), \tag{1}$$

where $x, h\in \mathbf{R}$ and $x, x+h\in \mathbf{I}$.

For concave function signs of inequality change their sense.

Function $f(x)=x^{m+1}$ is convex for: $m>0$ or $m<-1$, but concave for: $-1<m<0$, when $x>0$. Since $f'(x)=(m+1)x^m$, then for $h=1$, according to (1), we have

$$x^{m+1}+(m+1)x^m<(x+1)^{m+1}<x^{m+1}+(m+1)(x+1)^m. \tag{2}$$

Putting in (2) consecutively $x=1, 2, \dots, n-1$, and adding those inequalities, we find

$$\sum_{k=1}^{n-1} k^{m+1}+(m+1)\sum_{k=1}^{n-1} k^m<\sum_{k=2}^{n} k^{m+1}<\sum_{k=1}^{n-1} k^{m+1}+(m+1)\sum_{k=2}^{n} k^m. \tag{3}$$

The first inequality of expression (3) is equivalent to

$$\sum_{k=1}^{n} k^m<\frac{n^{m+1}-1}{m+1}+n^m, \tag{4}$$

and the second to

$$\frac{n^{m+1}-1}{m+1}+1<\sum_{k=1}^{n} k^m. \tag{5}$$

From (4) and (5) we obtain the inequalities

$$\frac{n^{m+1}-1}{m+1}+1<\sum_{k=1}^{n} k^m<\frac{n^{m+1}-1}{m+1}+n^m, \tag{6}$$

which are valid for $k\in \mathbf{N}$ and $m>0$, while the inequalities

$$\frac{n^{m+1}-1}{m+1}+n^m<\sum_{k=1}^{n} k^m<\frac{n^{m+1}-1}{m+1}+1, \tag{6a}$$

are valid for $k\in \mathbf{N}$ and $m<0$.

---

* Presented May 5, 1975 by P. M. VASIĆ.

In (6) and (6a) lower and upper limit for $m=-1$ become indefinite, but applying (1) to the concave function $f(x)=\log_a x$ $(a>1)$, we arrive at

$$\text{(6b)}\qquad \frac{\log_a (n\sqrt[n]{e}\,)}{\log_a e}<\sum_{k=1}^{n}\frac{1}{k}<\frac{\log_a (ne)}{\log_a e}\,.$$

In (6) – (6b) equality is valid for $n=1$.

Professor D. Nešić, who worked in Velika Škola (Beograd), discovered inequalities (6) in 1892, without giving conditions under which they are valid [**1**].

Inequalities (6) and (6a) can be considered as generalizations of some well-known particular inequalities. From (6a) for $m=-2$ we get

$$\text{(7)}\qquad \frac{1}{n^2}-\frac{1}{n}+1<\sum_{k=1}^{n}\frac{1}{k^2}<2-\frac{1}{n}\quad (n>1).$$

In [**2**], p. 47, we find the inequality

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<2.$$

Inequality (7) is more complete with stronger upper limit

For $m=-\frac{1}{2}$, (6a) becomes

$$\text{(8)}\qquad 2\sqrt{n}-2+\frac{1}{\sqrt{n}}<\sum_{k=1}^{n}\frac{1}{\sqrt{k}}<2\sqrt{n}-1\quad (n>1).$$

This inequality is contained in [**4**] as the second inequality in **3.1.12.** Approximation of the sum $\sum_{k=1}^{n} 1/\sqrt{k}$ is given in [**3**] with two inequalities, on page 110, in **2.1.10** and **2.1.11.**

Putting in (6b) $a=e$ it follows

$$\text{(9)}\qquad \frac{1}{n}+\log n<\sum_{k=1}^{n}\frac{1}{k}<1+\log n\quad (n>1).$$

In [**4**], p. 185, in **3.1.2**, the Schlömlich-Lemonnier inequality is quoted

$$\log (n+1)<1+\frac{1}{2}+\cdots+\frac{1}{n}<1+\log n.$$

Lower limit in (9) is stronger than the lower limit of this inequality.

REFERENCES

1. D. Nešić: *Dokaz obrasca* $\lim \frac{1^m+2^m+\cdots+(n-1)^m}{n^{m+1}}=\frac{1}{m+1}$. Glas Srpske akademije **33** (1892).

2. D. S. Mitrinović: *Važnije nejednakosti.* Beograd 1958.

3. D. S. Mitrinović: *Nejednakosti.* Beograd 1965.

4. D. S. Mitrinović (saradnik P. M. Vasić): *Analitičke nejednakosti.* Beograd 1970.