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531. SOME INEQUALITIES FOR TRIANGLE ELEMENTS*

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The student M. S. Jovanović has sumitted the paper entitled as above to the Editorial Committee. Ž. M. Mitrović and M. S. Stanković, who looked through the paper, have noticed that some results of Jovanović can be generalized by introducing a parameter *k*. The idea of Mitrović and Stanković has been carried out in this text.

Editorial Committee

Several inequalities for triangle elements, designated by capital letters starting from (A) are proved in this paper. The designations from the book **[1]** are used in this paper. Due to space, proofs of some inequalities are omitted.

(A) $$\sum\left(\frac{a}{r_a-r}\right)^k \geqq 3^{\frac{k}{2}+1} \qquad (k\geqq 1).$$

Equality holds for an equilateral triangle and $k=1$.

Proof. Since

(1) $$r_a-r=4R\sin^2\frac{\alpha}{2} \;\Rightarrow\; \frac{a}{r_a-r}=\operatorname{cotg}\frac{\alpha}{2},$$

we have

(2) $$\sum\left(\frac{a}{r_a-r}\right)^k=\sum\operatorname{cotg}^k\frac{\alpha}{2}.$$

Let us consider the function

(3) $$f(x)=\operatorname{cotg}^k x \quad \left(0<x<\frac{\pi}{2},\ k\geqq 1\right)$$

and its second derivative

$$f''(x)=\frac{k+\cos 2x}{\sin^4 x}\operatorname{cotg}^{k-2}x>0.$$

Hence, function f is convex, so that

(4) $$\sum_{i=1}^{3}\operatorname{cotg}^k x_i\geqq 3\operatorname{cotg}^k\frac{1}{3}(x_1+x_2+x_3).$$

If we put in (4) $x_1=\frac{\alpha}{2}$, $x_2=\frac{\beta}{2}$, $x_3=\frac{\gamma}{2}$, we get

(5) $$\sum\operatorname{cotg}^k\frac{\alpha}{2}\geqq 3\operatorname{cotg}^k\frac{\pi}{6}=3^{\frac{k}{2}+1}.$$

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From (2) and (5) (**A**) follows.

(B) $$\sum\left(\frac{r_b+r_c}{r_a-r}\right)^k \geqq 3^{k+1} \qquad \left(k \geqq \frac{1}{2}\right).$$

Equality holds for an equilateral triangle and $k=1/2$.

Proof. On the basis of (1) and

(6) $$r_b+r_c=4R\cos^2\frac{\alpha}{2},$$

we get

(7) $$\frac{r_b+r_c}{r_a-r}=\operatorname{cotg}^2\frac{\alpha}{2} \Rightarrow \sum\left(\frac{r_b+r_c}{r_a-r}\right)^k=\sum\operatorname{cotg}^{2k}\frac{\alpha}{2}.$$

Since, in virtue of (5),

(8) $$\sum\operatorname{cotg}^{2k}\frac{\alpha}{2}\geqq 3^{k+1},$$

from (7) and (8) we obtain (**B**).

(C) $$\sum\left(\frac{a^2}{r_b+r_c}\right)^k \geqq 3R^k \qquad (k\geqq 1).$$

Equality holds for an equilateral triangle and $k=1$.

Proof. Since, on the basis of (6), $\dfrac{a^2}{r_b+r_c}=4R\sin^2\dfrac{\alpha}{2}$, we have

(9) $$\sum\left(\frac{a^2}{r_b+r_c}\right)^k=(4R)^k\sum\sin^{2k}\frac{\alpha}{2}.$$

Due to (see [**1**]) $\sum\sin^2\dfrac{\alpha}{2}\geqq\dfrac{3}{4}$, we have

(10) $$\sum\sin^{2k}\frac{\alpha}{2}\geqq 3\left(\frac{1}{3}\sum\sin^2\frac{\alpha}{2}\right)^k\geqq\frac{3}{4^k}.$$

From (9) and (10), (**C**) follows.

(D) $$\sum\left(\frac{a^2}{r_a-r}\right)^k \leqq 3(3R)^k \qquad \left(k\leqq\frac{1}{2}\right).$$

Equality holds for an equilateral triangle and $k=1/2$.

Proof. From (1) we have $\dfrac{a^2}{r_a-r}=4R\cos^2\dfrac{\alpha}{2}$, i. e.,

(11) $$\sum\left(\frac{a^2}{r_a-r}\right)^k=(4R)^k\sum\cos^{2k}\frac{\alpha}{2}.$$

Let us consider the function

(12) $$f(x)=\cos^{2k}x \quad \left(0<x<\frac{\pi}{2},\ k\leqq\frac{1}{2}\right).$$

Since

$$f''(x)=2k(2k\sin^2 x-1)\cos^{2k-2}x<0,$$

we conclude that f is concave and we have

$$\sum_{i=1}^{3} \cos^{2k} x_i \leqq 3 \cos^{2k} \frac{1}{3}(x_1+x_2+x_3). \tag{13}$$

If we put $x_1=\frac{\alpha}{2}$, $x_2=\frac{\beta}{2}$, $x_3=\frac{\gamma}{2}$ in (13), we obtain:

$$\sum \cos^{2k}\frac{\alpha}{2} \leqq 3 \cos^{2k}\frac{\pi}{6}=\left(\frac{\sqrt{3}}{2}\right)^{2k}. \tag{14}$$

From (11) and (14) we get (**D**).

(**E**) $$\sum\left(\frac{a}{r_b+r_c}\right)^k \geqq 3^{1-\frac{k}{2}} \qquad (k\geqq 1).$$

Equality holds for an equilateral triangle and $k=1$.

Proof. On the hasis of (6) is $\frac{a}{r_b+r_c}=\operatorname{tg}\frac{\alpha}{2}$, i. e.

$$\sum\left(\frac{a}{r_b+r_c}\right)^k=\sum \operatorname{tg}^k\frac{\alpha}{2}. \tag{15}$$

Due to ([**1**]) $\sum \operatorname{tg}\frac{\alpha}{2}\geqq\sqrt{3}$, we have

$$\sum \operatorname{tg}^k\frac{\alpha}{2}\geqq 3\left(\frac{1}{3}\sum \operatorname{tg}\frac{\alpha}{2}\right)^k\geqq 3^{1-k/2}. \tag{16}$$

From (15) and (16) (**E**) follows.

(**F**) $$\sum\left(\frac{h_b+h_c}{r_b+r_c}\right)^k \leqq 3 \qquad (0<k\leqq 1).$$

Equality holds for an equilateral triangle and $k=1$.

Proof. Since

$$h_b+h_c=8R\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}, \tag{17}$$

from (6) and (17) we have

$$\frac{h_b+h_c}{r_b+r_c}=2\sin\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}\leqq 2\sin\frac{\alpha}{2}. \tag{18}$$

Analogously to (18) we get

$$\frac{h_c+h_a}{r_c+r_a}\leqq 2\sin\frac{\beta}{2}, \qquad \frac{h_a+h_b}{r_a+r_b}\leqq 2\sin\frac{\gamma}{2}. \tag{19}$$

From (18) and (19) we get:

$$\sum\left(\frac{h_b+h_c}{r_b+r_c}\right)^k\leqq 2^k\sum\sin^k\frac{\alpha}{2}. \tag{20}$$

Let us consider the function

$$f(x)=\sin^k x \quad \left(0<x<\frac{\pi}{2},\ 0<k\leqq 1\right). \tag{21}$$

Since $f''(x)=k(k\cos^2 x-1)\sin^{k-2}x<0$, function f is concave, so that we have:

$$\sum_{i=1}^{3}\sin^k x_i\leqq 3\sin^k\frac{1}{3}(x_1+x_2+x_3). \tag{22}$$

If we put in (22) $x_1=\frac{\alpha}{2}$, $x_2=\frac{\beta}{2}$, $x_3=\frac{\gamma}{2}$, we get

$$\sum \sin^k\frac{\alpha}{2}\leqq 3\sin^k\frac{\pi}{6}=\frac{3}{2^k}. \tag{23}$$

From (20) and (23) we get (F).

(G) $$\sum\left(\frac{h_b+h_c}{r+r_a}\right)^k\leqq 3\left(\frac{3}{2}\right)^k \qquad \left(0<k\leqq\frac{1}{2}\right).$$

Equality holds for an equilateral triangle and $k=1/2$.

Proof. From

$$r+r_a=4R\sin\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2} \tag{24}$$

and (17) it follows:

$$\frac{h_b+h_c}{r+r_a}=2\cos^2\frac{\alpha}{2}\ \Rightarrow\ \sum\left(\frac{h_b+h_c}{r+r_a}\right)^k=2^k\sum\cos^{2k}\frac{\alpha}{2}. \tag{25}$$

On the basis of (14) and (25) we get inequality (**G**).

(H)

1° $\prod(r_b+r_c)\leqq 27R^3$,

2° $\prod\frac{r_a-r}{r_b+r_c}\leqq\frac{1}{27}$,

3° $\prod\frac{h_b+h_c}{r_a+h_a}\leqq 1$,

4° $\prod\frac{r_a+h_a}{b+c}\leqq\frac{3}{8}\sqrt{3}$.

In 1°, 2°, 3°, 4° equality holds if the triangle is equilateral.

Proof. 1° Since $r_b+r_c=4R\cos^2\frac{\alpha}{2}$, we have

$$\prod(r_b+r_c)=(4R)^3\prod\cos^2\frac{\alpha}{2}\leqq 27R^3.$$

2° From (1) and (6) we get

$$\frac{r_a-r}{r_b+r_c}=\mathrm{tg}^2\frac{\alpha}{2}\ \Rightarrow\ \prod\frac{r_a-r}{r_b+r_c}=\prod\mathrm{tg}^2\frac{\alpha}{2}\leqq\frac{1}{27}.$$

3° Since

(26) $$r_a+h_a=4R\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\cos\frac{\beta-\gamma}{2},$$

from (17) and (26) it follows:

$$\frac{h_b+h_c}{r_a+h_a}=2\frac{\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$$

i. e.

$$\prod\frac{h_b+h_c}{r_a+h_a}=8\prod\sin\frac{\alpha}{2}\leqq 1.$$

4° Since

(27) $$b+c=4R\cos\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2},$$

on the basis of (24) we get

$$\frac{r_a+h_a}{b+c}=\frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}\Rightarrow\prod\frac{r_a+h_a}{b+c}=\prod\cos\frac{\alpha}{2}\leqq\frac{3}{8}\sqrt{3}.$$

(I) $$\sum\left(\frac{r+r_a}{b+c}\right)^k\leqq 3^{1-\frac{k}{2}}\quad(k\geqq 1).$$

Equality holds for an equilateral triangle and $k=1$.

Proof. From (24) and (27) we get:

(28) $$\frac{r+r_a}{b+c}=\mathrm{tg}\frac{\alpha}{2}\Rightarrow\sum\left(\frac{r+r_a}{b+c}\right)^k=\sum\mathrm{tg}^k\frac{\alpha}{2}.$$

From (28) and (16) we get inequality **(I)**.

(J) $$\sum\frac{r_a+h_a}{r+r_a}\geqq\frac{9}{2}.$$

Equality holds for an equilateral triangle.

Proof. From (24) and (26) we get:

$$\frac{r_a+h_a}{r+r_a}=\frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}\Rightarrow\sum\frac{r_a+h_a}{r+r_a}=\sum\frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}.$$

Since (see **[1]**)

$$\sum\frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}\geqq\frac{9}{2},$$

we obtain inequality (J).

(K)

1° $\sum\left(\frac{r_a+r}{h_a-r}\right)^k \geqq 3\cdot 2^k \quad (k\geqq 1)$,

2° $\sum\left(\frac{r_a-r}{h_a-2r}\right)^k \geqq 3\cdot 6^k \quad (k\geqq 1)$,

3° $\prod \frac{h_a-r}{r_b+r_c} \leqq \frac{1}{27}$.

Equality holds for an equilateral triangle and $k=1$.

Proof. 1° Since

$$h_a-r=4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\beta-\gamma}{2}, \tag{29}$$

on the basis of (24) we get

$$\frac{r_a+r}{h_a-r}=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} \Rightarrow \sum\left(\frac{r_a+r}{h_a-r}\right)^k=\sum\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}\right)^k. \tag{30}$$

Since

$$\sum\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}\right)^k \geqq 3\left(\frac{1}{3}\sum\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}\right)^k=3\left[\frac{1}{3}\sum\left(\operatorname{ctg}\frac{\beta}{2}\operatorname{ctg}\frac{\gamma}{2}-1\right)\right]^k,$$

and due to ([1]) $\sum\operatorname{ctg}\frac{\beta}{2}\operatorname{ctg}\frac{\gamma}{2}\geqq 9$, we obtain

$$\sum\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}\right)^k \geqq 3\cdot 2^k. \tag{31}$$

Inequality in 1° follows from (30) and (31).

(L) $$\sum\left(\frac{b+c}{h_a-r}\right)^k \geqq 3\cdot 12^{k/2} \quad (k\geqq 1).$$

Equality holds for the equilateral triangle and $k=1$.

(M) $$\sum\left(\frac{r_a}{h_a+2r_a}\right)^k \geqq 3^{1-k} \quad (k\geqq 1).$$

Equality holds for an equlateral triangle and $k=1$.

Proof. Using

$$h_a+2r_a=8R\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2} \tag{32}$$

and (26), we find

$$\frac{h_a+r_a}{h_a+2r_a}=\frac{1}{2}\left(1+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}\right),$$

i. e.,

$$\sum\left(1-\frac{r_a}{h_a+2r_a}\right)=\frac{1}{2}\sum\left(1+\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}\right)=2,$$

i. e.

$$\sum\frac{r_a}{h_a+2r_a}=1.$$

Then $\sum\left(\frac{r_a}{h_a+2r_a}\right)^k\geqq 3\left(\frac{1}{3}\sum\frac{r_a}{h_a+2r_a}\right)^k=3^{1-k}.$

(N) $$\sum\left(\frac{h_a-2r}{h_a+2r_a}\right)^k\geqq 3^{1-2k}\quad\left(k\geqq\frac{1}{2}\right).$$

Equality holds if the triangle is equilateral and $k=1/2$.

(O) $$\sum\left(\frac{r_b+r_c}{h_a+2r_a}\right)^k\geqq 3\left(\frac{2}{3}\right)^k\quad(k\geqq 1).$$

Equality holds if the triangle is equilateral and $k=1$.

Proof. From (6) and (39) we have:

$$\frac{r_b+r_c}{h_a+2r_a}=\frac{1}{2}\left(\frac{\cos\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}\right)^2,$$

i. e.,

$$\sum\left(\frac{r_b+r_c}{h_a+2r_a}\right)^k=\frac{1}{2^k}\sum\left(\frac{\cos\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}\right)^{2k}\geqq\frac{3}{2^k}\prod\left(\sec\frac{\alpha}{2}\right)^{\frac{2k}{3}}\geqq 3\left(\frac{2}{3}\right)^k.$$

(P) $$\sum\frac{r+r_a}{h_a+2r_a}\leqq\frac{4}{3}.$$

Equality holds for an equilateral triangle.

Proof. From (24) and (32) we have:

$$\frac{r+r_a}{h_a+2r_a}=\frac{1}{2}\left(1-\operatorname{tg}^2\frac{\beta}{2}\operatorname{tg}^2\frac{\gamma}{2}\right),$$

i. e.

(33) $$\sum\frac{r+r_a}{h_a+2r_a}=\frac{1}{2}\sum\left(1-\operatorname{tg}^2\frac{\beta}{2}\operatorname{tg}^2\frac{\gamma}{2}\right).$$

Since

(34) $$\sum\operatorname{tg}^2\frac{\beta}{2}\operatorname{tg}^2\frac{\gamma}{2}\geqq\frac{1}{3}\left(\sum\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}\right)^2=\frac{1}{3},$$

from (33) and (34) we get the inequality **(P)**.

(Q) $$\prod \frac{h_a - r}{b+c} \leqq \frac{\sqrt{3}}{72}.$$

Equality holds for an equilateral triangle.

Proof. From (27) and (29) we have:

(35) $$\frac{h_a - r}{b+c} = \frac{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} \Rightarrow \prod \frac{h_a - r}{b+c} = \prod \operatorname{tg}\frac{\alpha}{2}\sin\frac{\alpha}{2}.$$

Since ([**1**]) $\prod \sin\frac{\alpha}{2} \leqq \frac{1}{8}$, $\prod \operatorname{tg}\frac{\alpha}{2} \leqq \frac{\sqrt{3}}{9}$, we have

(36) $$\prod \sin\frac{\alpha}{2} \operatorname{tg}\frac{\alpha}{2} \leqq \frac{\sqrt{3}}{72}.$$

From (35) and (36) we get the inequality (**Q**).

(R) 1° $\sum \left(\frac{h_a - r}{h_a + r_a}\right)^k \geqq 3^{1-k}$ $(k \geqq 1)$,

2° $\prod \frac{h_a - r}{h_a + r_a} \leqq \frac{1}{27}$.

Equality holds for an equilateral triangle and $k = 1$.

(S) $$\sum \left(\frac{w_a}{h_b + h_c}\right)^k \geqq \left(\frac{r}{4R}\right)^{k/3} \quad (k > 0).$$

Equality holds for an equilateral triangle and $k = 1$.

(T) $$\sum \frac{w_a}{h_a + 2r_a} \geqq 1.$$

Equality holds for an equilateral triangle.

(U) $$\sum \left(\frac{w_a}{h_a - 2r}\right)^k \geqq 3^{k+1} \quad (k \geqq 1).$$

Equality holds for an equilateral triangle and $k = 1$.

REFERENCE

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