UNIV. BEOGRAD PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 498—№ 541 (1975), 159—161.

528. PRODUCTS OF HERMITIAN TRANSFORMATION*

Ali R. Amir-Moéz

Let A and B be hermitian transformations on E_n, a unitary space of dimension n, where at least one of A and B is nonnegative (positive semi-definite). Then proper values of AB are real. A generalization of minimax theorem for AB is given and other problems are suggested.

Introduction. Many inequalities for singular values of the product of two matrices have been obtained by R. C. THOMPSON, for example [**5**], and other papers of his. These inequalities are also valid for the proper values of the product of two non-negative transformations on E_n [**1**]. We shall not go into that. In this article we study the product of some hermitian transformations on E_n.

1. Notations. We shall consider a unitary space E_n of dimension n. Vectors will be denoted by Greek letters and complex numbers by italic small letters. Other notations will follow the standard ones.

2. Theorem. *Let A and B be hermitian transformations on E_n, where at least one of A and B is non-negative. Then proper values of AB are real.*

The proof is very simple. For example, let A be non-negative. Then $\sqrt{A}\,B\,\sqrt{A}$ is hermitian and has the same proper values as AB.

Since AB, in general, does not have an orthonormal set of proper vectors we shall study a geometric structure of proper vectors of AB in the next few sections.

3. Relatively orthogonal sets. Let H be a hermitian transformation on E_n. A set $\{\xi_1, \ldots, \xi_k\}$ is called *relatively orthogonal*, relative to H, if $(H\xi_i, \xi_j)=0$ for $i\neq j$.

4. Theorem. *Let H be a hermitian transformation on E_n with z zero, p positive and $n-(p+z)$ negative proper values. Let $\{\xi_1, \ldots, \xi_n\}$ be linearly independent and relatively orthogonal, relative to H. Then ξ's may be ordered in such a way that*

$$(H\xi_i, \xi_i)=0, \quad i=1, \ldots, z;$$
$$(H\xi_i, \xi_i)>0, \quad i=z+1, \ldots, z+p;$$
$$(H\xi_i, \xi_i)<0, \quad i=z+p+1, \ldots, n.$$

This theorem is due to SYLVESTER (Law of Inertia).

5. Theorem. *Let $\{\xi_1, \ldots, \xi_k\}$ be a set of relatively orthogonal vectors relative to a hermitian transformation H on E_n such that $(H\xi_i, \xi_i)\neq 0$, $i=1, \ldots, k$. Then the set is linearly independent.*

The proof will be omitted.

---

* Presented June 12, 1975 by D. S. MITRINOVIĆ.

6. Theorem. *Let A and B be hermitian transformations on E_n, where B is non-negative. Let $c_i \neq c_j$ be two proper values of AB. By 3, we know that c_i and c_j are real. Let*

$$AB\gamma_i = c_i \gamma_i, \quad AB\gamma_j = c_j \gamma_j, \quad \gamma_i \neq \vec{0}, \quad \gamma_j \neq \vec{0}.$$

Then $(B\gamma_i, \gamma_j) = 0$.

Proof. The proof follows the pattern of a hermitian transformation. One observes that

$$(AB\gamma_i, B\gamma_j) = c_i(\gamma_i, B\gamma_j) = (B\gamma_i, AB\gamma_j) = c_j(B\gamma_i, \gamma_j).$$

7. Corollary. *Let A and B satisfy hypotheses of 6. Then the set of proper vectors of AB is linearly independent.*

8. Theorem. *Let H be a non-singular positive hermitian transformation on E_n. Let $\{\xi_1, \ldots, \xi_n\}$ be a set of relative orthogonal vectors relative to H. Then this set can be normalized to $\{\alpha_1, \ldots, \alpha_n\}$ such that $(H\alpha_i, \alpha_j) = \delta_{ij}$. Thus the set $\{\alpha_1, \ldots, \alpha_n\}$ will be called relatively orthonormal relative to H.*

The proof is quite simple. One may extend the theorem to the case that H is singular.

9. Components of a vector. Let $\{\alpha_1, \ldots, \alpha_n\}$ be relatively orthonormal relative to a non-singular positive transformation H on E_n. Then $\xi \in E_n$ can be written as

$$\xi = \sum_{i=1}^{n} (\xi, H\alpha_i)\alpha_i.$$

Also one obtains that

$$(H\xi, \xi) = \sum_{i=1}^{n} |(\xi, H\alpha_i)|^2.$$

10. A minimax principle. *Let A and B be hermitian transformations on E_n and B be positive. Let $c_1 \geqq \cdots \geqq c_n$* be proper values *of $C = AB$. Then*

$$c_k = \sup_{\substack{M \\ \dim M = k}} \inf_{\substack{\xi \in M \\ (B\xi, \xi) = 1}} (AB\xi, B\xi).$$

The proof follows step by step the techniques of Fischer's minimax theorem. We give an outline of the proof.

Proof. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of relative orthonormal proper vectors of AB such that $AB\alpha_i = c_i\alpha_i$, $i = 1, \ldots, n$. Let $M = [\alpha_1, \ldots, \alpha_k]$. Then for $\xi \in M$ & $(B\xi, \xi) = 1$ we have

$$(AB\xi, B\xi) = \sum_{i=1}^{k} c_i(B\alpha_i, \alpha_i) \geqq c_k. \tag{1}$$

On the other hand let M be a k-dimensional subspace of E_n. Let $N = [\alpha_k, \ldots, \alpha_n]$. Then there exists $\xi \in M \cap N$ such that $(B\xi, \xi) = 1$. Thus

$$(AB\xi, \xi) = \sum_{i=k}^{n} c_i(B\alpha_i, a_i) \leqq c_k. \tag{2}$$

Comparing (1) and (2) the proof is complete.

Indeed, this theorem can be generalized in many directions as was done in [**1**], [**2**], [3], and [**4**], ... Thus they can be assigned as exercises.

One can change B to a non negative transformation and obtain modified results. We omit that.

Since in the formula, $(AB\xi, B\xi)$ is a quadratic form in $B\xi$, one may obtain inequalites containing proper values of A, AB, and B. We omit the details.

REFERENCES

1. A. R. Amir-Moéz: *Extreme properties of eigenvalues of a hermitian transformation and singular values of the sum and product of linear transformations.* Duke Math. J. **23** (1956), 463—476.
2. A. R. Amir-Moéz, G. E. Johnston: *Extreme properties of products of quadratic form.* Duke Math. J. **37** (1970), 61—65.
3. Ky. Fan: *Maximum properties and inequalities for the eigenvalues of completely continuos operators.* Proc. Nat. Acad. Sci. USA **37** (1951), 760—766.
4. M. Marcus, J. L. McGregor: *Extremal properties of hermitian matrices.* Canad. J. Math. **8** (1956), 524—531.
5. R. C. Thompson, S. Therianos: *The eigenvalues and singular valuls of matrix sums and products* VIII: *Displacements of Indices.* Aequationes Math. **7** (1972), 219—242.

Department of Mathematics
Texas Tech. University
Lubbock, TX. 79409
USA