UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 498—№ 541 (1975), 49—50.

506. EVALUATION OF A CLASS OF DEFINITE INTEGRALS*

M. L. Glasser

Let $\Phi(x)$ be a polynomial, trigonometric or exponential function of x. In this paper we examine the class of integrals

$$\int_{-\infty}^{+\infty} \frac{\Phi(x)}{\prod_{j=1}^{N} \operatorname{hyp}(n_j x + a_j)}\, dx$$

where hyp_j denotes the hyperbolic sine or cosine, the n_j are any N integers and the a_j are any N constants (which may be complex). Integrals of this form with $N=2$ arise in transport theory and isolated examples have been evaluated by a variety of techniques [1]. In this note we shall illustrate by an example how a general member of the class may be expressed in finite terms and explore some consequences (For the case $N=1$ see [2]).

Consider

$$I=\int_{-\infty}^{+\infty} \frac{e^{-\alpha x}\, dx}{\cosh x \cosh (x+a) \cosh (x+b)} \qquad (a \neq b \neq 0,\ |\operatorname{Re} \alpha|<3)$$

The integrand has simple poles at $x=\frac{1}{2}\pi i,\ \frac{1}{2}\pi i-a,\ \frac{1}{2}\pi i-b$ with residues $ie^{-i\pi\alpha/2} \operatorname{csch} a \operatorname{csch} b,\ ie^{-i\pi\alpha/2} e^{\alpha a} \operatorname{csch} a \operatorname{csch}(a-b),\ ie^{-i\pi\alpha/2} e^{\alpha b} \operatorname{csch} b \operatorname{csch}(b-a)$ respectively. Let σ denote their sum. By integrating around the contour extending from $-\infty$ to $+\infty$ along the real axis and $+\infty$ to $-\infty$ along the line $\operatorname{Im} x=\pi i$, since $\operatorname{hyp}(u+i\pi)=-\operatorname{hyp} u$, we see that $(1+e^{-\pi i\alpha}) I=2\pi i\sigma$. Hence

$$\text{(1)} \qquad \mathrm{I}=\frac{\pi}{\cos\left(\frac{\pi\alpha}{2}\right)}\left[\frac{e^{\alpha a}}{\sinh a \sinh (b-a)}+\frac{e^{\alpha b}}{\sinh b \sinh (a-b)}-\frac{1}{\sinh a \sinh b}\right].$$

This result, which is valid for real values of the parameter, can be extended by analytic continuation. By operating on both sides of (1) with $\Phi(D_\alpha)\ (D_\alpha=d/d\alpha)$ and taking the limt $\alpha \to 0$, we replace $e^{-\alpha x}$ by $\Phi(x)$; by taking α to be imaginary and equating real and imaginary parts, we obtain the trigonometric case.

From the inversion theorem for the two sided LAPLACE transform we find from (1)

---

* Presented April 26, 1975 by P. M. VASIĆ.

(2) sech x sech $(x+a)$ sech $(x+b)$

$$=\pi\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,\frac{e^{sx}}{\cos(\pi s/2)}\left(\operatorname{csch}(a-b)\left(\frac{e^{sb}}{\sinh b}-\frac{e^{sa}}{\sinh a}\right)-\operatorname{csch}a\operatorname{csch}b\right)\quad(0<c<1).$$

In this way a number of inverse MELLIN transforms may be evaluated. For example, in just the way (2) was obtained, we find

$$\text{(3)}\qquad \operatorname{sech}x\operatorname{sech}(x+t)=\operatorname{csch}t\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\cdot\frac{\pi}{\sin(\pi s/2)}\,e^{xs}(e^{st}-1).$$

And from (3) we have

$$\frac{1}{4}\operatorname{sech}^2\frac{x}{2}=\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,\frac{\pi s}{\sin\pi s}\,e^{xs}$$

which is the basic formula used in [3]. Some other interesting consequences of (3) are

$$\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,\frac{\pi}{\sin\pi s}\left(\psi(b+s)-\psi(c-s)\right)$$

$$=\frac{1}{4}\left(\psi(b)-\psi(c)+\psi\left(\frac{c+1}{2}\right)-\psi\left(\frac{c}{2}\right)+\psi\left(\frac{b}{2}\right)-\psi\left(\frac{b+1}{2}\right)\right)$$

where $\psi(x)=\frac{d}{dx}\log\Gamma(x)$, and

$$\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,\frac{\pi}{\sin\pi s}\log\left(\frac{s-b}{s-a}\right)=\frac{1}{4}\log\left(\frac{a}{b}\right)+\log\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{a+1}{2}\right)}.$$

REFERENCES

1. M. L. GLASSER and V. E. WOOD: *Evaluation of Some Transport Integrals*, II. J. Appl. Phys. **37** (1966), 4364—4366.
2. D. S. MITRINOVIĆ (saradnik J. D. KEČKIĆ): *Kompleksna analiza — Zbornik zadataka i problema*. Beograd 1972.
3. M. L. GLASSER: *Note of the evaluation of some Fermi integrals*. J. Math. Phys. **5** (1964), 1150—1152.

Department of Applied Mathematics
University of Waterloo,
Waterloo, Ont. N2L 3G1, Canada