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505. THE RECONSTRUCTION PROBLEM FOR CHARACTERISTIC POLYNOMIALS OF GRAPHS*

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Let $X=\{x_1, \ldots, x_n\}$ be the vertex set of a graph G. G_i denotes the subgraph of G induced by the set $X\setminus\{x_i\}$. Well-known ULAM's conjecture states that for $n>2$ the graph G can be reconstructed uniquely from the collection of its subgraphs $G_i (i=1, \ldots, n)$. It was proved that ULAM's conjecture is true for some classes of graphs (regular, disconnected, trees etc.), but in the general case the problem remains unsolved.

In this paper we shall pose and consider a problem of a similar kind. Let A be the adjacency matrix of the graph G with n vertices. The characteristic polynomial

(1) $$P_G(\lambda)=\det(\lambda I-A)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$$

of the adjacency matrix is called the characteristic polynomial of G. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of A form spectrum of G. The multiplicity of the number zero in the spectrum of G will be denoted by $\eta(G)$.

$P_{G_i}(\lambda)$ is the characteristic polynomial of the subgraph G_i and the collection of the polynomials $P_{G_i}(\lambda) (i=1, \ldots, n)$ will be denoted by $\mathcal{P}(G)$.

The problem reads:

Is it true, that for $n>2$ the characteristic polynomial $P_G(\lambda)$ of a graph G is determined uniquely by the collection of the characteristic polynomials $P_{G_i}(\lambda)$ $(i=1, \ldots, n)$ of subgraphs $G_i (i=1, \ldots, n)$?

If the answer to this question is positive for a graph G, we shall say that $P_G(\lambda)$ can be reconstructed $(\text{by } \mathcal{P}(G))$.

We shall first discuss the relation between this and ULAM's problem. It is known that a graph in general is not determined by its characteristic polynomial, i. e. there exist non-isomorphic graphs with the same characteristic polynomial (or spectrum). This means that the characteristic polynomial contains only partial information about the graph structure. Therefore, in our problem less is to be proved about G than in ULAM's problem, but also we have less data about G (only characteristic polynomials of subgraphs but not the subgraphs themselves). It is interesting to note that the positive answer to our problem would imply the validity of ULAM's conjecture for those graphs which are determined by their spectrum. Unfortunately, all nontrivial graphs, for which one knows at present that they are characterized by spectra, are regular, while ULAM's conjecture is trivially true for regular graphs. Therefore, in connection with the results from this paper it would be of interest to try to find some non-trivial classes of non-regular graphs which are characterized by their spectra.

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On the other hand, if the characteristic polynomial can be reconstructed (for a class of graphs) and if ULAM'S conjecture is not true (for the same class of graphs), then two non-isomorphic graphs with the same collection of subgraphs G_i $(i=1, \ldots, n)$, which would present a counterexample to ULAM's conjecture, would have the same spectra. Therefore, it is perhaps reasonable to search counterexamples to ULAM's conjecture among graphs which form pairs of isospectral non-isomorphic graphs.

We begin the considerations with a simple statement that was noted in [1], namely

$$P_G'(\lambda) = \sum_{i=1}^{n} P_{G_i}(\lambda). \tag{2}$$

Integrating this relation one gets all coefficients of the characteristic polynomial (1) of G except a_n. Thus only a_n is to be further determined.

Note that there are graphs having the same derivative of the characteristic polynomials but different coefficients a_n. The smallest example (for $n>2$; since in the case of $n=2$ our conjecture as well as ULAM's conjecture are not true) is provided by the graphs $G' = K_{1,3}$ (star) and $G'' = P_4$ (path).

The corresponding polynomials are $P_{G'}(\lambda) = \lambda^4 - 3\lambda^2$, $P_{G''}(\lambda) = \lambda^4 - 3\lambda^2 + 1$. But these graphs do not represent a counterexample to our conjecture since $\mathcal{P}(G')$ and $\mathcal{P}(G'')$ are different. We have examined several other such pairs of graphs failing to find a counterexample.

Now we shall go a step further.

Lemma 1. *If, besides $\mathcal{P}(G)$, an eigenvalue of G is known, $P_G(\lambda)$ can be reconstructed.*

The proof is obvious.

This simple lemma represents the starting point for further considerations. In some cases it is possible to find an eigenvalue on the basis of the collection $\mathcal{P}(G)$.

Lemma 2. *The collection of vertex degrees of G can be determined from $\mathcal{P}(G)$.*

Proof. The number m of edges of G is equal to $-a_2$, where a_2 is a coefficient from (1). Thus m is obtainable from $\mathcal{P}$. In a similar way the number m_i of edges of each of the subgraphs G_i $(i=1, 2, \ldots, n)$ can be obtained. The quantities $m - m_i$ $(i=1, 2, \ldots, n)$ are obviously the vertex degrees in G.

Theorem 1. *If the collection $\mathcal{P}$ for a graph G is known, it can be established whether or not G is regular. If G is regular, its characteristic polynomial can be reconstructed.*

Proof. The first statement follows from Lemma 2. According to this lemma, the degree of G can also be determined. But, the degree of a regular graph is always an eigenvalue of that graph. Then according to Lemma 1, $P_G(\lambda)$ can be reconstructed.

This completes the proof of the theorem.

This type of reasoning can be exploited in all cases when the collection of vertex degrees implies the existence of a certan eigenvalue of G (for example, when G has isolated vertices, the number zero is an eigenvalue of G).

Let A be a hermitian of order n and B a principal submatrix of A of order $n-1$. It is well known that eigenvalues of A and eigenvalues of B separate each other. Thus, if one of matrices A, B has an eigenvalue λ of multiplicity $p\,(p>1)$, the other matrix has the same eigenvalue λ with multiplicity at least $p-1$. This fact together with Lemma 1 implies the following theorem.

Theorem. 2. *If at least one polynomial of $\mathcal{P}(G)$ has a root with the multiplicity greater than* 1, *$P_G(\lambda)$ can be reconstructed. This is always true if G has an eigenvalue of mutiplicity greater than* 2.

Unfortunately, it seems that there are many graphs G for which all subgraphs G_i have only simple eigenvalues.

Let us consider now bipartite graphs. All subgraphs G_i must then be bipartite and, conversely, if all G_i are bipartite, G is also bipartite except when G is a circuit of odd length. In this last case all G_i are paths with $n-1$ vertices. The fact that G_i is bipartite or is a path can be recognized by $P_{G_i}(\lambda)$. The first statement follows from the well known theorem that bipartite graphs and only they have the spectra symmetric with respect to the zero point. The second statement follows from the fact that a path is characterized by its spectrum. Thus,

Lemma 3. *$\mathcal{P}(G)$ determines whether or not G is bipartite.*

If G is a bipartite graph with n_1 vertices of one colour and n_2 vertices o another colour, $n_1 \geqq n_2$, then inequality $\eta(G) \geqq n_1 - n_2$ holds [2]. Among the subgraphs G_i there exists necessarily one with n_1 vertices of one colour and n_2-1 vertices of another, and hence $\eta(G_i) \geqq n_1 - n_2 + 1$. If $n_1 \neq n_2$ we have $\eta(G_i) \geqq 2$ and according to Theorem 2, $P_G(\lambda)$ can be reconstructed.

Note that for odd n we always have $n_1 \neq n_2$.

According to all these facts, $P_G(\lambda)$ can be reconstructed from $\mathcal{P}(G)$ (without additional information) if G is bipartite except for the case $n_1 = n_2$ and $\eta(G)=0$ or $\eta(G)=2$ (for $n_1=n_2$, $\eta(G)$ is an even number).

It is known (see [3]) that the multiplicity of the number zero in the spectrum of a bipartite graph G without circuits of length $4s\,(s=1, 2, \ldots)$ is equal to $n-2t$, where t is the maximal number of mutually non-adjacent edges of G. Then it is clear that by removal of a vertex, incident to an edge from such a maximal system of non-adjacent edges, a subgraph G_i will be obtained, which has the maximal system of non-adjacent edges of smaller cardinality. Therefore, $\eta(G_i)$ is greater than $\eta(G)$. In such a way $P_G(\lambda)$ can be reconstructed in the described class of bipartite graphs also in the case of $n_1=n_2$ and $\eta(G)=2$.

We summarize these facts in the following theorem.

Theorem 3. *The reconstruction conjecture for the characteristic polynomial is true for all bipartite graphs G except, perhaps, for the case $n_1=n_2$ and $\eta(G)=0$ or, $\eta(G)=2$ and G has a circuit of length $4s\,(s \in \mathbf{N})$.*

Corollary. *For all trees with $\eta(G)>0$, $P_G(\lambda)$ can be reconstructed.*

If besides $\mathcal{P}(G)$ some additional information about G is known, we can, of course, say something more. For example, if we know that G is disconnected, we can find $P_G(\lambda)$. Actually, in this case the maximal root among the roots of all polynomials $P_{G_i}(\lambda)$ is the maximal eigenvalue of G.

Further, if we know that G is connected, $P_G(\lambda)$ can be reconstructed for all trees. This can be seen in the following way. From the knowledge of the number of edges and vertices one can in the case of connected graphs simply decide whether G is a tree or not. Now, because of the Corollary of Theorem 3 only trees with $\eta(G)=0$ are of interest here. In the case of trees the fact that $\eta(G)=0$ is obviously equivalent to $\eta(G_i)=1$ for all $i=1, 2, \ldots, n$. Hence from $\mathcal{P}(G)$ one can establish that $\eta(G)=0$ and for this case we have $a_n=(-1)^{n/2}$ [4].

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