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504. GRAPH EQUATION $L^n(G) = \overline{G}$ *

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In this paper we shall consider only finite, undirected graphs without loops or multiple edges, or shortly, according to HARARY [1], only graphs. For all definitions and notation the reader is referred to [1]. Here, we shall mention only the following definitions.

The complement $\overline{G}$ of a graph G is the graph having the same set of vertices as G and in which two (different) vertices are adjacent, if and only if they are not adjacent in G.

The line graph $L(G)$ of a graph G is the graph whose vertex set coincides with the edge set of G and in which two (different) vertices are adjacent, if the corresponding edges are adjacent in G.

$L^n(G)$ is defined in a natural way; $L^0(G) = G$ and $L^n(G) = L(L^{n-1}(G))$. Throughout this paper symbol $=$ will stand for an isomorphism between two graphs.

In literature there are many results which can be stated in the form of „graph equations". This notion was introduced in [2] by the following authors D. CVETKOVIĆ, I. LACKOVIĆ and S. SIMIĆ. In the same paper they have given the list of some examples. Here we shall mention only a few of them.

In [3], V. V. MENON has solved the graph equation $L(G) = G$. He has found that G is a solution to the above equation, if and only if G is a regular graph of degree two. Later, the same author in [4] has proved that the equation $L^n(G) = G$ has the same solutions for any n.

The following short and elegant result is due to M. AIGNER. In [5], he has proved that only the following two graphs from Fig.[1] 1 satisfy the equation $L(G) = \overline{G}$.

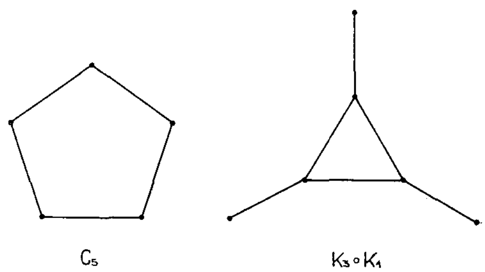

Fig. 1

In the present paper we shall generalize the result of M. AIGNER. Namely, we shall solve the equation $L^n(G) = \overline{G}$.

In order to solve the equation $L^n(G) = \overline{G}$ we shall consider two cases.

Case 1: G is a connected graph.

Let $\Delta(G)$ be the maximal vertex degree of G. Now we shall develop our further discussion according to $\Delta(G)$.

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* Presented March 24, 1975 by D. M. CVETKOVIĆ.

[1] The sign „$\circ$" on Fig. 1 denotes a corona of two graphs.

Suppose, first that $\Delta(G) \leqq 2$.

Now since G is a connected graph and since $\Delta(G) \leqq 2$, G must be a path or a cycle.

If G is a path P_m (m being the number of vertices in P_m) then $L^n(G)$ is the path P_{m-n} for $m>n$ or an empty graph[1] for $m \leqq n$. Since the complement of the path is neither a path nor the empty (except in the case when for $m=4$, $\overline{P}_4 = P_4$ holds), it follows immediately that G could not be a path for any n.

Assume that G is a cycle. Then $L^n(G) = G$ for each n. So we have to find all self-complementary cycles. Through a straightforward observations we notice that cycle at the length five is the sole self-complementary cycle. Indeed, the cycle C_5 is the solution to the equation $L^n(G) = \overline{G}$ for every n.

Suppose now that $\Delta(G) \geqq 3$.

Let us assume that G is a (p_0, q_0)-graph and also, that $L^k(G)$ is (p_k, q_k)-graph $(1 \leqq k \leqq n)$. It is clear that $p_k = q_{k-1}$ $(1 \leqq k \leqq n)$ holds. Also, $q_k \geqq p_k$ $(1 \leqq k \leqq n)$, since $L^k(G)$ is, of course, connected and has at least one cycle (the latter ensues from the fact that $\Delta(G) \geqq 3$). So we have that $p_k \geqq p_{k-1}$ for $1 \leqq k \leqq n$. Since $p_0 = p_n$ (because of $L^n(G) = \overline{G}$), it follows that $p_0 \geqq p_1 = q_0$ must be satisfied. So, G is now a tree or a unicyclic graph (different from a cycle).

Suppose that G is a unicyclic graph different from a cycle. Then $q_1 > p_1$ holds ($L(G)$ has at least two cycles) so that it follows immediately that in this case n may be equal only to one. Since the equation $L(G) = \overline{G}$ has been already solved, we shall not deal with that case.

It remains to consider the case when G is a tree. Then, having in view M. Ainger's result, $n \geqq 2$. Let us put that $L^{n-1}(G) = H$ and let us further consider the equation $\overline{L(H)} = G$ where G is a tree.

According to the results of D. Cvetković and S. Simić (see [7], Theorem 3) and having in view that $\Delta(G) \geqq 3$, it is not difficult to deduce that G could be any of the following graphs from Fig. 2.

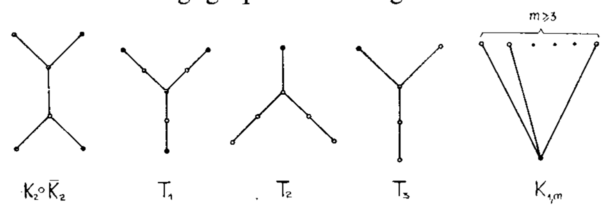

Fig. 2

It is easy to see (by a direct verification) that the first graph from Fig. 2 is a solution to the equation $L^2(G) = \overline{G}$. (Clearly, for $n>2$ it cannot be a solution to the equation $L^n(G) = \overline{G}$.)

The remaining graphs from Fig. 2 are not the solutions to the equation $L^n(G) = \overline{G}$ for any n. Namely, graphs T_1, T_2, T_3 from Fig. 2 are not the solutions to the equation $L^n(G) = \overline{G}$ because the pairs of graphs $L^n(T_i)$, $\overline{T}_i$ $(i=1,$

[1] The empty graph is a graph without vertices and of course without edges. On the advantages of introducing the notion of the empty graph, in general, see [6]. In this paper we shall treat the empty graph as an auxiliary instrument.

2, 3; $n\geqq 2$) do not have the same number of vertices (it can easily be seen) except in the case of T_3 and $n=3$, when it is not difficult to notice that $L^3(T_3)\neq\overline{T}_3$. It is evident that graph $K_{1,m}(m\geqq 3)$ is not a solution, since its complement is a disconnected graph.

Case 2: G is a disconnected graph.

Let us put again that $L^{n-1}(G)=H$. Then we have that $\overline{L(H)}=G$, G being a disconnected graph. Now, according to [7], it can be proved (the proof will be given in the Appendix) that G is one of the following graphs: lK_1 $(l\geqq 2)$, $3K_2, 2K_2$, $(K_{n_1,n_2}-pK_2)\cup K_1$[1)] $\big(p=\min(n_1,n_2);\ p\geqq 0,\ n_1,n_2\geqq 1\big)$; the last graph is given on Fig. 3(a).

Now it is only a routine to see that none of graphs lK_1 $(l\geqq 2)$, $3K_2, 2K_2$ can be the solution to the equation $L^n(G)=\overline{G}$ for any n. In the case when we have $G=(K_{n_1,n_2}-pK_2)\cup K_1$, the situation is slightly complicated. Namely, since $H\big(=L^{n-1}(G)\big)$ is also equal to $L^{-1}(\overline{G})$, it can easily be found that if G is a graph from Fig. 3(a) then H is a graph from Fig. 3(b) where q is an arbitrary nonnegative integer.

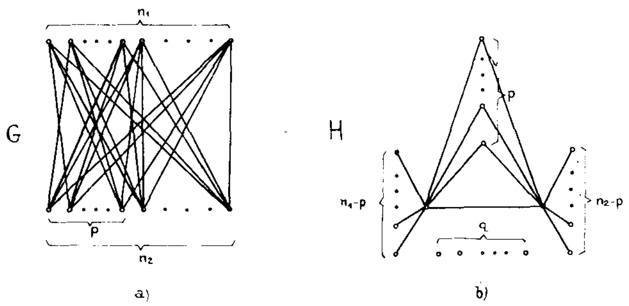

Fig. 3

We shall, of course, consider the case when $n\geqq 2$. Clearly, the graph H from Fig. 3 must be a line graph. But then, because of L. Beineke's forbidden induced subgraph $K_{1,3}$ for a graph H, it follows that $n_1\leqq 2$, $n_2\leqq 2$. Now, it can be easily verified that the rest of graphs $(K_{n_1,n_2}-pK_2)\cup K_1$ do not satisfy the equation $L^n(G)=\overline{G}$ for any n.

So we have proved the following theorem.

Theorem. *The equation $L^n(G)=\overline{G}$ has only the following solutions:*

(i) *for* $n=1$, $G=C_5$ *or* $G=K_3\circ K_1$ (*result of M. Aigner*),

(ii) *for* $n=2$, $G=C_5$ *or* $G=K_2\circ\overline{K_2}$,

(iii) *for* $n\geqq 3$, $G=C_5$.

Appendix

Now, we shall give the proof of the fact already used in the text; namely, we shall prove the following theorem.

1) The graph $K_{n_1,n_2}-pK$ is described in [7]. It is obtained from K_{n_1,n_2} by deleting p independent edges.

Theorem. *If G is a disconnected graph such that $G=\overline{L(H)}$ for some graph H, then G is one of the following graphs:*

(i) lK_1 $(l \geqq 2)$,

(ii) $3K_2$,

(iii) $2K_2$,

(iv) $(K_{n_1,n_2}-pK_2) \cup K_1$ $(p \leqq \min(n_1, n_2);\ p \geqq 0;\ n_1, n_2 \geqq 1)$.

Proof. In order to avoid trivial cases we shall assume that G has at least one edge. In that case it is easy to prove that G is bichromatic graph. Namely, if G is a disconnected graph having at least one odd cycle, then according to Lemma 2 (see [7]) that cycle must be C_3 or C_5. But then, having in view that G is a disconnected graph, one of the graphs $C_3 \cup K_1$ or $C_5 \cup K_1$ (both are the complements of L. Beineks's forbidden induced subgraphs for the line graphs) would appear in G, so it immediately follows that G is bichromatic. Now according to Lemma 1 (see [7]) G can have 2 or 3 components. Since bichromatic graphs G with the property that $G=\overline{L(H)}$ for some graph H are completely described in [7], we immediately get the conclusions of the theorem.

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