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501. ON COEFFICIENTS OF THE GREGORY FORMULA*

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The mutual connection of the known results on coefficients g_n of the Gregory formula

$$\frac{1}{h}\int_{x}^{x+mh} f(t)\,\mathrm{d}t=\sum_{k=0}^{m} f(x+kh)+\sum_{n=1}^{+\infty} g_n\{(-1)^n\Delta^{n-1}f(x)-\nabla^{n-1}f(x+mh)\}$$

is presented in this paper. The known asymptotic formula for g_n is improved. An algorithm for quick calculation of coefficients g_n is proposed and the corresponding computer program is developed.

First ten coefficients g_n have the values:

$$g_1=\frac{1}{2},\quad g_2=\frac{1}{12},\quad g_3=\frac{1}{24},\quad g_4=\frac{19}{720},\quad g_5=\frac{3}{160},\quad g_6=\frac{863}{60480},$$

$$g_7=\frac{275}{24192},\quad g_8=\frac{33953}{3628800},\quad g_9=\frac{8183}{1036800},\quad g_{10}=\frac{3250433}{479001600}.$$

Coefficients g_n were calculated by T. CLAUSSEN for $n\leqq 13$, K. PEARSON for $n\leqq 14$, R. A. FISHER—F. YATES for $n\leqq 17$, A. N. LOWAN—H. E. SALZER for $n\leqq 20$. H. T. DAVIS gives

$$g_{20}=0.00256702255,\qquad g_{100}=0.0002974763.$$

Integration of the GREGORY—NEWTON interpolation formula

$$f(x+nh)=(1+\Delta)^n f(x)$$

yields the GREGORY formula, so that

$$g_n=(-1)^{n+1}\int_0^1\binom{x}{n}\mathrm{d}x. \tag{1}$$

Result (1) was obtained by J. W. L. GLAISHER [see WHITTAKER—ROBINSON 166. See also MILNE 196, BAHVALOV 166, KRYLOV—ŠUL'GINA 61].

Substituting $x=-s$ from (1) it follows that

$$g_n=(-1)^{n+1}\int_{-1}^{0}\binom{-s}{n}\mathrm{d}s,$$

[see PHILLIPS—TAYLOR 132].

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From (1), substituting $t=1-x$, it follows that

$$g_n = -\int_0^1 \binom{n-2+t}{n} \mathrm{d}t, \tag{2}$$

[compare with NIELSEN 4].

Starting from (1) it is possible to derive the generating function for the coefficients g_n

$$\sum_{n=1}^{+\infty} g_n t^n = \sum_{n=1}^{+\infty} \left((-1)^{n+1} \int_0^1 \binom{x}{n} \mathrm{d}x \right) t^n = \int_0^1 \left(\sum_{n=1}^{+\infty} (-1)^{n+1} \binom{x}{n} t^n \right) \mathrm{d}x$$

$$= \int_0^1 \left(1-(1-t)^x\right) \mathrm{d}x = \left(x - \frac{(1-t)^x}{\log(1-t)} \right) \Bigg|_{x=0}^{x=1}$$

i.e.,

$$1 + \frac{t}{\log(1-t)} = \sum_{n=1}^{+\infty} g_n t^n \quad (|t| \leqq 1). \tag{3}$$

[See: KUNZ 170—171, ISAACSON—KELLER 318, BOOLE 55, MINEUR 183].

Using (3) and the development

$$-\log(1-t) = \sum_{k=1}^{+\infty} \frac{1}{k} t^k \quad (|t|<1)$$

the known recurrent expression

$$\sum_{k=1}^{n} \frac{1}{n+1-k} g_k = \frac{1}{n+1} \tag{4}$$

is obtained. [See, for example: KELLY 57, BEREZIN—ŽITKOV 266].

Formula (4) is one of the developments of the n-th order determinant, quoted by WHITTAKER—ROBINSON 130

$$g_n = (-1)^{n+1} \begin{vmatrix} \frac{1}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \\ \vdots & & & & \end{vmatrix}.$$

Starting from

$$\frac{t}{(t-1)\log(1-t)} = \sum_{n=0}^{+\infty} c_n t^n, \quad \sum_{k=0}^{n} \frac{c_k}{n+1-k} = 1,$$

P. HENRICI 253—254 gives $g_n = c_n - c_{n-1}$.

R. V. HAMMING 149 presents the calculation of g_n by the method of undetermined coefficients.

Starting from the EULER—MACLAURIN formula

$$\frac{1}{h}\int_{x}^{x+nh} f(t)\,\mathrm{d}t=\sum_{m=0}^{n} f(m)-\frac{f(0)+f(n)}{2}-\sum_{k=1}^{+\infty}\frac{h^{2k-1}B_{2k}}{(2k)!}\left(f^{(2k-1)}(n)-f^{(2k-1)}(0)\right)$$

where B_{2k} are BERNOULLI's numbers, and from

$$\left(h\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f(x)=\left(\log(1+\Delta)\right)^k f(x),$$

the GREGORY formula is obtained, so that this is another way for calculating the coefficients g_n. See: SCHEID 117, HILDEBRAND 202, RALSTON 135, BOOTH 182, JEFFREYS—SWIRLES 45.

The relationship between the coeficients g_n with BERNOULLI's numbers of first order

$$g_n=\frac{(-1)^n B_n^{(n-1)}}{n!\,(n-1)}.$$

and with BERNOULLI's polynomials

$$g_n=\frac{(-1)^n}{n!}B_n^{(n)}(1)$$

are known. [See FLETCHER—MILLER—ROSENHEAD—COMRIE 108].

H. T. DAVIS has, on the basis of a formula equivalent to (2), and the expression

$$\Gamma(x)\,\Gamma(1-x)=\frac{\pi}{\sin\pi x}$$

derived the formula

$$g_n=\frac{1}{\pi}\int_0^1\frac{\Gamma(1+s)\,\Gamma(n-s)}{\Gamma(n+1)}\sin\pi s\,\mathrm{d}s,$$

wherefrom, by means of the approximative formula

(5) $$\frac{\Gamma(n+x)}{\Gamma(n)}\sim n^x$$

he obtained the result

(6) $$g_n\sim\frac{\Gamma(1+\xi)}{n(\log^2 n+\pi^2)}\qquad(0\leqq\xi\leqq 1,\ n\to+\infty).$$

Notice that inequality

$$g_n<\frac{1}{n(\log^2 n+\pi^2)}$$

holds for $n\geqq 14$.

We shall now give an improvement of formula (6). From (2) it follows that

$$g_n = \frac{1}{n} \int_0^1 \frac{(t-t^2)\,\Gamma(n+t-1)}{\Gamma(1+t)\,\Gamma(n)}\,\mathrm{d}t,$$

wherefrom, by means of the approximation (5) we get

$$g_n \sim \frac{1}{\Gamma(1+\theta)} \frac{1}{n \log^2 n} \qquad (0 \leqq \theta \leqq 1, \quad n \to +\infty). \tag{7}$$

Results (6) and (7) can be harmonized if the values of the gamma function are equal to unity, so that

$$g_n \sim \frac{1}{n \log^2 n} \qquad (n \to +\infty). \tag{8}$$

From all the mentioned formulas only (4) is suitable for direct computer calculation of g_n but for small values of n, only. When g_n is calculated by means of (2), all previous coefficients $g_1, g_2, \ldots, g_{n-1}$ participate, which leads to error accumulation. One of the summands is $1/(n+1)$, i.e. it is considerably greater than the result g_n which unfailingly provokes further decrease in accuracy. Finally, calculation time for the coefficient g_n is proportional to n, and that of all $g_1, g_2, \ldots, g_n$ is proportional to n^2.

We propose an algorithm where only a few coefficients $A(k, n)$ participate in the formation of g_n, so that the time needed for the calculation of the table of values $g_1, g_2, \ldots, g_n$ is proportional to n. From (1) it follows that

$$g_n = (-1)^{n+1} \int_{-1/2}^{1/2} \binom{t+1/2}{n} \mathrm{d}t = -\frac{1}{n!} \int_{-1/2}^{1/2} \prod_{k=1}^{n} \left(\frac{2n-2k-1}{2} - t\right) \mathrm{d}t$$

wherefrom

$$g_n = \int_{-1/2}^{1/2} \sum_{k=1}^{n+2} A(k, n)\, t^{k-2}\,\mathrm{d}t$$

where

$$\begin{aligned} &A(k, 0) = 0 \quad (k > 2), \quad A(2, 0) = -1, \quad A(1, n) = 0, \\ &A(k, n) = A(k, n-1) - \frac{1}{n}\left\{\frac{3}{2} A(k, n-1) + A(k-1, n-1)\right\}. \end{aligned} \tag{9}$$

Coefficients g_n are

$$g_n = \sum_{m=1}^{[(n+2)/2]} \frac{A(2m, n)}{(2m-1)\, 4^{m-1}}$$

where $x \mapsto [x]$ designates the function „integral part of x". Since the modulus of $A(k, n)$ decreases very rapidly with the increase of k in the calculation of g_n, it is sufficient to use the formula

$$g_n \approx \sum_{m=1}^{[L/2]} \frac{A(2m, n)}{(2m-1)\, 4^{m-1}}.$$

Dimension L of the auxiliary vector A is determined from the sufficient condition $L \leqq 20+D$, where D is the greatest number of the accurate decimal digits of the computer. Fig. 1 displays the computer realisation of the program GREGO, which calculates $G(N)$ for $N=1\,(1)\,M$. For the variable L (dimension of vector A) the value 30 is assumed, since the program is intended for the computer operating with 10 significant digits at most. For the computer with a relative error 10^{-30} it is sufficient to assume $L=50$.

From (9) it follows that

$$A(2,\ n)=\frac{\pi^{-1/2}}{2n-1}\,\frac{\Gamma\left(n+\dfrac{1}{2}\right)}{\Gamma(n+1)},$$

wherefrom we get

$$A(2,\ n)\sim\frac{1}{2\sqrt{\pi}}\left(n-\frac{1}{4}\right)^{-3/2}$$

because

$$\Gamma(n+1)/\Gamma\left(n+\frac{1}{2}\right)\sim\sqrt{n+\frac{1}{4}}$$

is valid. Compare MITRINOVIĆ—VASIĆ 281.

Table 1 contains some values of coefficients g_n and c_n, related by $g_n=1/(n(\log^2 n+\log n+c_n))$. The last decimal digit of the numbers in Table 1 should not be considered as certain.

n	g_n	c_n
1	0.5000000000	2.00
2	0.0833333333	4.83
4	0.0263888889	6.17
8	0.0093565366	6.96
16	0.0034973499	7.41
32	0.0013509659	7.66
64	0.0005346403	7.77
128	0.0002157724	7.81
256	0.0000885505	7.82
512	0.0000368751	7.81
1024	0.0000155557	7.80
2048	0.0000066381	7.80
4096	0.0000028619	7.80
8192	0.0000012453	7.82
16384	0.0000005463	7.84

Table 1

```
      SUBROUTINE GREGO(M,G)
      DIMENSION A(30),G(1)
      DATA L/30/
      DO 1 K=1,L
    1 A(K)=0
      A(2)=-1
      G(1)=-A(2)/2
      R=3*G(1)
      DO 5 N=1,M
      S=-N
      K=L
    2 A(K)=(A(K)*R+A(K-1))/S+A(K)
      K=K-1
      IF(K-1) 3,3,2
    3 K=L
      S=0
    4 S=S/4+A(K)/(K-1)
      K=K-2
      IF(K) 5,5,4
    5 G(N)=S
      RETURN
      END
```

Fig. 1

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D. S. MITRINOVIĆ, S. M. JOVANOVIĆ, D. Đ. TOŠIĆ and J. D. KEČKIĆ have read this paper in manuscript and have made some valuable remarks and suggestions.

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