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499. ON INEQUALITIES FOR $\Gamma(x+1)/\Gamma(x+1/2)$*

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1. Interesting well-known inequalities for

$$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}=\frac{1}{\sqrt{\pi}}\frac{(2n)!!}{(2n-1)!!}=\frac{2}{\sqrt{\pi n}}\frac{2^{2n}}{B(n,n)}=\frac{1}{\sqrt{\pi}}\frac{2^{2n}(n!)^2}{(2n)!}=\frac{1}{\sqrt{\pi}}\frac{2^{2n}}{\binom{2n}{n}}$$

will be dealt with in the present paper. Apart from that, sharper inequalities for the function $x\mapsto\Gamma(x+1)/\Gamma\left(x+\frac{1}{2}\right)$ will be proposed and its integral representation will be given

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}=\sqrt{x}\ \exp\left\{\sum_{k=1}^{n}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}-\int_0^{+\infty}\left(\frac{\operatorname{th}t}{2t}-\sum_{k=1}^{n}\frac{2^{2k}(2^{2k}-1)B_{2k}}{2\cdot(2k)!}t^{2k-2}\right)e^{-4tx}\,dt\right\}.$$

2. From the general result of W. GAUTSCHI [1]

$$n^{1-s}\leqq\frac{\Gamma(n+1)}{\Gamma(n+s)}\leqq(n+1)^{1-s}\qquad(0\leqq s\leqq 1)$$ [10, 281]

it follows for $s=1/2$

(1) $$\sqrt{n}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n+1}.$$

J. WALLIS [2] has given a double inequality

(2) $$\sqrt{n}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n+\frac{1}{2}}.$$ [4, 83]

Using FIHTENGOLC's result

$$\frac{(2n)!!}{(2n-1)!!}=\sqrt{\pi n}\ \exp\frac{4\theta-\theta'}{4n}\qquad(0<\theta<1,\quad 0<\theta'<1)$$ [3, 371]

inequality

(3) $$\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n}\ \exp\frac{1}{6n}$$

is immediately proved.

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The following inequality is also known

$$\left(\frac{2\pi n(2n+1)}{4n+1}\right)^{1/2}<\frac{(2n)!!}{(2n-1)!!}, \qquad [\mathbf{4}, 82]$$

see [**4**], p. 322, i.e.,

(4)
$$\sqrt{n+\frac{1}{4}-\frac{1}{16n+4}}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}.$$

Designating by $B(a, b)$ the betta function, J. T. CHU [**5**] gives the following result

$$\sqrt{\frac{4n-3}{4n-2}}<2^{2n-1}\sqrt{\frac{2n-1}{2\pi}}\;B(n, n)<\sqrt{\frac{4n-2}{4n-1}}, \qquad [\mathbf{10}, 283]$$

which is equivalent to

(5)
$$\sqrt{n+\frac{1}{4}-\frac{1}{(4n-2)^2}}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n+\frac{1}{4}+\frac{1}{16n-4}}.$$

D. K. KAZARINOFF [**6**] has proved

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}}<\frac{(2n-1)!!}{(2n)!!}<\frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \qquad [\mathbf{10}, 190]$$

i.e.,

(6)
$$\sqrt{n+\frac{1}{4}}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n+\frac{1}{2}}.$$

G. N. WATSON [**7**] gives a more general result

(7)
$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\leqq\sqrt{x+\frac{1}{\pi}} \qquad (x\geqq 0), \qquad [\mathbf{10}, 281]$$

wherein the constants $1/4$ and $1/\pi$ are the best possible.

The upper bounds in (6) and (7) lag behind the upper bound in (5).

J. GURLAND [**8**] has improved the quoted results

$$\frac{4n+3}{(2n+1)^2}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2<\pi<\frac{4}{4n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2, \qquad [\mathbf{4}, 83]$$

i.e.,

(8)
$$\sqrt{n+\frac{1}{4}}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n+\frac{1}{4}+\frac{1}{16n+12}}.$$

J. T. CHU [**5**] has given two more representations of inequality (8).

Finally, A. V. BOYD [9] has quated even better result

$$\left(n+\frac{1}{4}+\frac{1}{32n+32}\right)^{1/2}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\frac{n+\frac{1}{2}}{\left(n+\frac{3}{4}+\frac{1}{32n+48}\right)^{1/2}} \qquad [\mathbf{10}, 281]$$

i.e.,

$$(9) \qquad \sqrt{n+\frac{1}{4}+\frac{1}{32n+32}}<\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}<\sqrt{n+\frac{1}{4}+\frac{1}{32n-\frac{64n-148}{8n+11}}}.$$

3. On the basis of the functional inequality

$$\Gamma(x+1)=x\Gamma(x)$$

the implications

$$(10) \qquad f(x)\leqq\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \Rightarrow \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\leqq\frac{x+\frac{1}{2}}{f\left(x+\frac{1}{2}\right)},$$

$$(11) \qquad \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\leqq g(x) \Rightarrow \frac{x}{g\left(x-\frac{1}{2}\right)}\leqq\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$$

are valid. By means of the results (10) and (11) it is possible, on the basis of the one-fold inequality for the function $x\mapsto\frac{\Gamma(x+1)}{\Gamma(x+1/2)}$, make a double inequality of the corresponding accuracy. Such inequalities are (2), (5), (8) and (9). On the basis of (11), and adopting

$$g(x)=\sqrt{x+\frac{1}{4}+\frac{1}{32x+8}},$$

we get

$$(12) \qquad \sqrt{x+\frac{1}{4}+\frac{1}{32x+8+\frac{36}{4x-1}}}<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x+\frac{1}{4}+\frac{1}{32x+8}}.$$

Inequality (12) is sharper than all the above quoted.

4. On the basis of LEGENDRE's formula

$$2^{z-1}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right)=\sqrt{\pi}\,\Gamma(z)$$

and also on the basis of the well-known result

$$\operatorname{Log}\Gamma(z)=\left(z-\frac{1}{2}\right)\log z-z+\log\sqrt{2\pi}+\int_0^{+\infty}\left(\frac{1}{2}-\frac{1}{t}+\frac{1}{e^t-1}\right)\frac{e^{-tz}}{t}\,dt \qquad (\operatorname{Re} z>0)$$

the equality

$$(13)\qquad \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}=\sqrt{x}\exp\left\{\int_0^{+\infty}\frac{\operatorname{th} t}{2t}e^{-4tx}\,dt\right\}\qquad (x>0)$$

can be obtained. Since

$$\int_0^{+\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{2\cdot(2k)!}t^{2k-2}e^{-4tx}\,dt=\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}$$

the inequality

$$(14)\qquad \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}=\sqrt{x}\exp\left\{\sum_{k=1}^{n}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right.$$
$$\left.+\int_0^{+\infty}\left(\frac{\operatorname{th} t}{2t}-\sum_{k=1}^{n}\frac{2^{2k}(2^{2k}-1)B_{2k}}{2\cdot(2k)!}t^{2k-2}\right)e^{-4tx}\,dt\right\}$$

follows. Since the sign of the sub-integral value and the sign for $(-1)^n$ are equal, the following inequality is valid

$$(15)\qquad \sqrt{x}\exp\sum_{k=1}^{2m}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}<\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x}\exp\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}},$$

where m and l are natural numbers and $x>0$. Inequalities in (15) are sharper than (12).

*

D. S. Mitrinović and P. M. Vasić have read this article in manuscript and made some valuable remarks and suggestions.

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