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497. A CONTRIBUTION TO YOUNG'S INEQUALITY*

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Editorial Committee

1. Let f be a real, continuous and increasing function on $[0, c]$, where $c>0$. If $f(0)=0$, $a\in[0, c]$, $b\in[0, f(c)]$ then

$$\int_0^a f(x)\mathrm{d}x+\int_0^b f^{-1}(x)\,\mathrm{d}x\geqq ab. \tag{1}$$

Inequality (1) is the well known YOUNG's inequality. This paper discusses the upper bound of

$$F(a, b, f)=\int_0^a f(x)\,\mathrm{d}x+\int_0^b f^{-1}(x)\,\mathrm{d}x.$$

2. Let f be a real valued function which satisfies the conditions:

a) f is continuous on $[0, c]$, $c>0$,

b) f is increasing on $[0, c]$,

c) $f(0)=0$,

and let us also suppose that

d) $a\in[0, c]$ and $b\in[0, f(c)]$,

e) $G=G(a, b)$ is a real function of arguments a and b,

f) for every a and b, $G(a, b)>0$.

In this paper we will prove two theorems.

Theorem 1. *There is no function G which satisfies the conditions* e) *and* f) *such that the inequality*

$$F(a, b, f)\leqq G(a, b) \tag{2}$$

should be valid for each function F satisfying the conditions a), b), c) *and for each a and b satisfying the condition* d).

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Proof. We shall prove that for each a and b and for each given function G it is possible to construct a function f such that the inequality (2) is not true. Suppose that a, b and that the function G are given. The function f defined by $f(x)=\big(G(a,b)+1\big)(e^x-1)$ satisfies the conditions a), b), c), and we have

(3) $$\int_0^a f(x)\,\mathrm{d}x=\big(G(a,b)+1\big)(e^a-a).$$

Because of the inequality $e^a-a\geqq 1$, from (3) follows

(4) $$\int_0^a f(x)\,\mathrm{d}x\geqq G(a,b)+1>G(a,b).$$

From (4) follows $F(a,b,f)>G(a,b)$ which was to be proved.

Lemma 1. *Suppose that the conditions* a), b), c), d) *are satisfied. Hence*

(5) $$af(a)\geqq bf^{-1}(b)$$

is equivalent to

(6) $$f(a)\geqq b.$$

Proof. Let (6) be true. From the monotonicity of the function f^{-1}, (6) implies

(7) $$f^{-1}\big(f(a)\big)\geqq f^{-1}(b) \qquad \text{i.e.} \quad a\geqq f^{-1}(b).$$

From (6) and (7) follows (5). Let (5) be true. Suppose that

(8) $$f(a)<b.$$

From the monotonicity of the function f^{-1}, from (8) follows

(9) $$a<f^{-1}(b).$$

From (8) and (9) follows $af(a)<bf^{-1}(b)$ which contradicts (5). Hence, from (5) follows (6).

Lemma 2. *Suppose that the conditions* a), b), c), d) *are satisfied. Then*

(10) $$\int_0^a f(x)\,\mathrm{d}x+\int_0^{f(a)} f^{-1}(x)\,\mathrm{d}x=af(a)$$

and

(11) $$\int_0^b f^{-1}(y)\,\mathrm{d}y+\int_0^{f^{-1}(b)} f(y)\,\mathrm{d}y=bf^{-1}(b).$$

If the conditions of the above Lemma are satisfied then the functions $\int_0^u f(x)\,\mathrm{d}x$ and $\int_0^v f^{-1}(x)\,\mathrm{d}x$ are continuous convex functions, complementary in YOUNG's sense (see for example [1] or [2]). For the proof of Lemma 2 see [2].

Theorem 2. *Suppose that the conditions* a), b), c), d) *are satisfied. Then the following inequality holds*

$$F(a, b, f) \leqq \max\left(af(a),\ bf^{-1}(b)\right). \tag{12}$$

Proof. If $f(a) \geqq b$ then $\int_0^b f^{-1}(y)\,dy \leqq \int_0^{f(a)} f^{-1}(y)\,dy$ and the inequality (12) follows from the following

$$F(a, b, f) \leqq \int_0^a f(x)\,dx + \int_0^{f(a)} f^{-1}(x)\,dx = af(a).$$

In the case when we have $f(a) \leqq b$ the proof follows by interchanging a and b and f and f^{-1}. This completes the proof of the theorem.

EXAMPLES. The functions $f(x) = x^{p-1}$ $(p > 1)$ and $g(x) = \log(1 + x)$ satisfy the conditions of our theorems for $x \geqq 0$. By substitution of these two functions in the inequality (12) we obtain the following two inequalities

$$\frac{a^p}{p} + \frac{b^q}{q} \leqq \max(a^p,\ b^q), \quad \frac{1}{p} + \frac{1}{q} = 1$$

and

$$(1+a)\log(1+a) - (1+a) + e^b - b \leqq \max\left(a\log(1+a),\ b(e^b - 1)\right)$$

where $a, b \geqq 0$.

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