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495. THE GENERALIZATION OF AN INEQUALITY FOR A FUNCTION AND ITS DERIVATIVES

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In monograph [**1**, p. 362] the following result is given:

Let function f be defined in an interval (a, b). Let us assume that there exists f''' and that it is an increasing function in the interval (a, b). Then, if $x \in (a+1, b-1)$, we have

$$f''(x) < f(x+1) - 2f(x) + f(x-1). \tag{1}$$

Let us introduce an operator Δ by means of

$$\Delta^n f(x) = \Delta^{n-1} f(x+1) - \Delta^{n-1} f(x), \quad \Delta^0 f(x) = f(x) \qquad (n \in N).$$

It may be shown that

$$\Delta^n f(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x+n-i). \tag{2}$$

In this paper we shall use the following denotation

$$\lambda_n = \Delta^n f(x-k) - f^{(n)}(x) \qquad (n \in N), \tag{3}$$

where $k = \left[\frac{n}{2}\right]$, by which the inequality (1) has the form

$$0 < \lambda_2.$$

Lemma 1. *If* $0 \leqq m \leqq n$, *then*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x+n-i)^m = \begin{cases} 0 & (0 \leqq m < n) \\ n! & (m = n). \end{cases}$$

Proof. Let $x \mapsto x^m$ $(0 \leqq m \leqq n)$. Then, using (2),

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x+n-i)^m = \Delta^n (x^m).$$

Since

$$\Delta^n (x^m) = \begin{cases} 0 & (0 \leqq m < n) \\ n! & (m = n) \end{cases}$$

(see [**2**]), the proof is completed.

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Let the function f be defined and be $(n+1)$-times differentiable in (a, b). Let us introduce the operators T and R by means of

$$T(n, l; f)=\sum_{m=0}^{n} \frac{f^{(m)}(x)}{m!} l^m \quad \text{and} \quad R(n, l; f)=\frac{l^{n+1}}{(n+1)!} f^{(n+1)}(x+l\theta_l),$$

where $0 \leq \theta_l \leq 1$. Then the development of the function f into the TAYLOR expansion in the neighbourhood of point $x \in (a, b)$, is given by

$$f(x+l)=T(n, l; f)+R(n, l; f) \qquad (x+l \in (a, b)).$$

Lemma 2. *Equality*

$$\lambda_n=(-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} R(n, j-k; f) \tag{4}$$

is valid.

Proof. Since

$$f\big(x+(n-k-i)\big)=T(n, n-k-i; f)+R(n, n-k-i; f),$$

using (2), we have

$$\Delta^n f(x-k)= \sum_{i=0}^{n} (-1)^i \binom{n}{i} T(n, n-k-i; f)+ \sum_{i=0}^{n} (-1)^i \binom{n}{i} R(n, n-k-i; f).$$

Since

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} T(n, n-k-i; f)= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{m=0}^{n} \frac{f^{(m)}(x)}{m!} (n-k-i)^m$$

$$= \sum_{m=0}^{n} \frac{f^{(m)}(x)}{m!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-k-i)^m,$$

using Lemma 1, it follows

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} T(n, n-k-i; f)=f^{(n)}(x). \tag{5}$$

According to (5), the equality (3) becomes

$$\lambda_n= \sum_{i=0}^{n} (-1)^i \binom{n}{i} R(n, n-k-i; f). \tag{6}$$

Placing in (6) $i=n-j$, we obtain (4), and thus the proof is completed.

For a sequence of functions $F=(F_1, F_2, \ldots, F_k)$, let us define the terms $D(F)$ and $G(F)$ as follows:

$$D(F)= \sum_{i=1}^{\left[\frac{k}{2}\right]} F_{k-2i+1}, \qquad G(F)= \sum_{i=0}^{\left[\frac{k-1}{2}\right]} F_{k-2i}.$$

In further discussion we shall define the upper and the lower limit for λ_n, under condition that $f^{(n+1)}$ is a nondecreasing function.

We shall distinguish the cases when n is even and when n is odd.

1. *Case* $n=2k$. Let sequence F be defined by

$$F_m=F_m(k, x; f)=\binom{2k}{k-m}m^{2k+1}[f^{(2k+1)}(x+m)-f^{(2k+1)}(x-m)] \quad (m=1, 2, \dots, k).$$

Theorem 1. *If* $f^{(2k+1)}$ *is a nondecreasing function in* (a, b) $(b-a>2k)$, *then*

$$-\frac{D(F)}{(2k+1)!}\leqq\lambda_{2k}\leqq\frac{G(F)}{(2k+1)!} \qquad (a+k<x<b-k;\ k\in N). \tag{7}$$

Proof. Since $R(2k, 0; f)=0$, we have

$$\lambda_{2k}=\sum_{j=0}^{2k}(-1)^j\binom{2k}{j}R(2k, j-k; f)$$
$$=\frac{1}{(2k+1)!}\sum_{j=0}^{k-1}(-1)^j\binom{2k}{j}(k-j)^{2k+1}[f^{(2k+1)}(x+\theta_{k-j}(k-j))$$
$$-f^{(2k+1)}(x-\theta_{k-j}(k-j))],$$

i.e.

$$\lambda_{2k}=\frac{1}{(2k+1)!}\sum_{m=1}^{k}(-1)^{k-m}s_m,$$

where

$$s_m=\binom{2k}{k-m}m^{2k+1}[f^{(2k+1)}(x+\theta_m m)-f^{(2k+1)}(x-\theta_{-m}m)].$$

Since $f^{(2k+1)}$ is a nondecreasing function in (a, b), $0\leqq\theta_m\leqq 1$ and $0\leqq\theta_{-m}\leqq 1$, we deduce that $s_m\in[0, F_m]$ when $x\in(a+m, b-m)$.

Upon summing up the intervals (see [3]), we obtain

$$\lambda_{2k}\in I \qquad \forall x\in(a+k, b-k),$$

where the interval I is given by

$$I=\frac{1}{(2k+1)!}\sum_{m=1}^{k}(-1)^{k-m}[0, F_m]$$
$$=\frac{1}{(2k+1)!}[-(F_{k-1}+F_{k-3}+\cdots),\ F_k+F_{k-2}+\cdots]$$
$$=\left[-\frac{D(F)}{(2k+1)!},\ \frac{G(F)}{(2k+1)!}\right].$$

Thus, Theorem 1 is proved.

EXAMPLE 1. If f''' is a nondecreasing function in (a, b), (7) is reduced to

$$0\leqq\lambda_2\leqq\frac{1}{3!}F_1 \qquad \forall x\in(a+1, b-1),$$

i.e.

$$f''(x)\leqq f(x+1)-2f(x)+f(x-1)\leqq f''(x)+\frac{1}{6}\big(f'''(x+1)-f'''(x-1)\big). \tag{8}$$

The first inequality in (8) includes inequality (1).

Similarly, for $k=2$ and $k=3$, (7) is reduced respectively to

$$-\frac{1}{30}\big(f^{(5)}(x+1)-f^{(5)}(x-1)\big)\leqq\lambda_4\leqq\frac{4}{15}\big(f^{(5)}(x+2)-f^{(5)}(x-2)\big)$$

and

$$-\frac{16}{105}\left(f^{(7)}(x+2)-f^{(7)}(x-2)\right)\leqq\lambda_6\leqq\frac{243}{560}\left(f^{(7)}(x+3)-f^{(7)}(x-3)\right)$$
$$+\frac{1}{336}\left(f^{(7)}(x+1)-f^{(7)}(x-1)\right).$$

2. *Case* $n=2k+1$. Let us define sequences $P=(P_1, P_2, \ldots, P_k)$ and $Q=(Q_1, Q_2, \ldots, Q_k)$ by

$$P_m=P_m(k, x; f)=m^{2k+2}\left[\binom{2k+1}{k-m+1}f^{(2k+2)}(x)+\binom{2k+1}{k-m}f^{(2k+2)}(x-m)\right],$$

$$Q_m=Q_m(k, x; f)=P_m(k, x+m; f) \qquad (m=1, 2, \ldots, k).$$

Theorem 2. *If* $f^{(2k+2)}$ *is a nondecreasing function in* (a, b) $(b-a>2k+1)$, *then*

(9) $$\frac{(k+1)^{2k+2}f^{(2k+2)}(x)+D(P)-G(Q)}{(2k+2)!}\leqq\lambda_{2k+1}$$
$$\leqq\frac{(k+1)^{2k+2}f^{(2k+2)}(x+k+1)+D(Q)-G(P)}{(2k+2)!}$$

$$(a+k<x<b-k-1;\ k\in N)$$

and

$$\frac{1}{2}f''(x)\leqq\lambda_1\leqq f''(x+1) \qquad (a<x<b-1).$$

The proof of Theorem 2 is similar to that of Theorem 1.

EXAMPLE. If f'' is nondecreasing function in (a, b), then

$$f'(x)+\frac{1}{2}f''(x)\leqq f(x+1)-f(x)\leqq f'(x)+\frac{1}{2}f''(x+1) \qquad \forall\, x\in(a, b-1).$$

Note that similar inequality is given in [1]. Namely, if f is an increasing function, the inequality

$$f'(x)<f(x+1)-f(x)<f'(x+1).$$

is proved.

For $k=1$ and $k=2$, (9) is reduced to

$$\frac{5}{8}f^{(4)}(x)-\frac{1}{8}f^{(4)}(x+1)\leqq\lambda_3\leqq\frac{2}{3}f^{(4)}(x+1)-\frac{1}{8}f^{(4)}(x)-\frac{1}{24}f^{(4)}(x-1)$$

and

$$\frac{1}{144}f^{(6)}(x-1)+\frac{15}{16}f^{(6)}(x)-\frac{4}{9}f^{(6)}(x+2)\leqq\lambda_5$$
$$\leqq\frac{81}{80}f^{(6)}(x+3)+\frac{1}{72}f^{(6)}(x)-\frac{4}{45}f^{(6)}(x-1).$$

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