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469. ON AN INEQUALITY FOR CONVEX FUNCTIONS*

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In this article we shall prove an inequality for twice differentiable convex functions and weighted arithmetical means which in special case reduces to the well known Hadamard's inequality.

1. In literature the inequalities

$$f\left(\frac{a+b}{2}\right) \leqq \frac{1}{b-a} \int_a^b f(t)\, dt \leqq \frac{f(a)+f(b)}{2} \tag{1}$$

which are valid for arbitrary convex functions are known. As far as we know these inequalities appeared for the first time in 1883 (see [**1**]) under the assumptions that f is twice differentiable function on $[a, b]$ such that $f''(t) \geqq 0$ for all $t \in [a, b]$. Under the above assumptions for function f in [**5**] an elementary proof of inequalities (1) is given. The right inequality in (1) is known as HADAMARD's inequality (see [**2**] pp. 12—16). For convex functions (namely, without assumptions of differentiability, but only with the assumption that the quoted function is JENSEN convex and continuous) in paper [**4**] an inequality more general than inequality (1) is proved. In literature there are many generalisations and applications of inequality (1) (see, for example [3], page 180—183).

2. It is natural to consider the following two problems in connection with the above inequality (1):

1° Does there exist an u, $v \in [a, b]$ such that the ratio $\frac{1}{v-u} \int_u^v f(t)\, dt$ lies between $f\left(\frac{a+b}{2}\right)$ and $\frac{f(a)+f(b)}{2}$?

2° For which u and v the following inequalities

$$f\left(\frac{pa+qb}{p+q}\right) \leqq \frac{1}{v-u} \int_u^v f(t)\, dt \leqq \frac{pf(a)+qf(b)}{p+q} \tag{2}$$

are valid, if the weights p and q are given positive numbers?

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We will give an answer to these two questions, assuming that the function f is twice differentiable on $[a_1, b_1]$ such that we have $f''(t) \geqq 0$ for all $t \in [a_1, b_1]$.

Let us start with LAGRANGE's interpolation formulae

$$f(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) + \frac{1}{2} f''(t)(x-a)(x-b)$$

where $a_1 \leqq a \leqq t \leqq b \leqq b_1$. By integration this equality from u to v, we obtain

$$\int_u^v f(x)\,dx = \frac{(v-b)^2-(u-b)^2}{2(a-b)} f(a) + \frac{(v-a)^2-(u-a)^2}{2(b-a)} f(b) + \frac{1}{2} f''(t_1) \int_u^v (x-a)(x-b)\,dx. \tag{3}$$

In order to reduce the right side of equality (3) to $\frac{pf(a)+qf(b)}{p+q}(v-u)$ it is sufficient that

$$(v-b)^2-(u-b)^2 = \frac{2p(v-u)(a-b)}{p+q}, \tag{4}$$

$$(v-a)^2-(u-a)^2 = \frac{2q(v-u)(b-a)}{p+q}, \tag{5}$$

$$\int_u^v (x-a)(x-b)\,dx \leqq 0. \tag{6}$$

System of equalities (4) and (5) can be reduced to one equality, from which we get

$$u+v = \frac{2pa+2qb}{p+q}. \tag{7}$$

If we introduce the notation

$$\frac{pa+qb}{p+q} = A \tag{8}$$

then the condition (6) becomes

$$uv \geqq 4A^2 - 3(a+b)A + 3ab. \tag{9}$$

In virtue of (7) and (8) it follows that $v = 2A - u$, so that from (9) we obtain

$$u^2 - 2uA + 4A^2 - 3(a+b)A + 3ab \leqq 0. \tag{10}$$

Necessary and sufficient condition that inequality (10) has a solution is that

$$D \equiv -12A^2 + 12(a+b)A - 12ab \geqq 0.$$

Therefrom we get

$$a \leqq A \leqq b \quad \text{i. e.} \quad a \leqq \frac{pa+qb}{p+q} \leqq b,$$

which is always satisfied. From (10) we then have

$$\frac{pa+qb}{p+q} - \frac{b-a}{p+q}\sqrt{3pq} \leqq u \leqq \frac{pa+qb}{p+q} + \frac{b-a}{p+q}\sqrt{3pq} \tag{11}$$

and from (7) and (11) it follows that

$$\frac{pa+qb}{p+q}-\frac{b-a}{p+q}\sqrt{3pq}\leqq v\leqq\frac{pa+qb}{p+q}+\frac{b-a}{p+q}\sqrt{3pq}. \tag{12}$$

If we put

$$u-\frac{pa+qb}{p+q}=x, \tag{13}$$

then from (7) we obtain

$$v=\frac{pa+qb}{p+q}-x, \tag{14}$$

so that conditions (11) and (12) reduce to the condition

$$-\frac{b-a}{p+q}\sqrt{3pq}\leqq x\leqq\frac{b-a}{p+q}\sqrt{3pq}. \tag{15}$$

In virtue of above obtained results we can conclude that if conditions (15) are fulfilled then also conditions (4), (5) and (6) are satisfied, such that the inequality

$$\frac{1}{2x}\int_{A-x}^{A+x} f(t)\,\mathrm{d}t\leqq\frac{pf(a)+qf(b)}{p+q} \tag{16}$$

follows from (3), where x satisfies condition (15).

If we substitute $a=A-x$ and $b=A+x$ in the first inequality of inequalities (1) we obtain

$$f\left(\frac{pa+bq}{p+q}\right)\leqq\frac{1}{2x}\int_{A-x}^{A+x} f(t)\,\mathrm{d}t. \tag{17}$$

On the basis of (16) and (17) we get the following theorem.

Theorem 1. *If f is twice differentiable convex function in the interval $[a_1, b_1]$ then the following inequalities*

$$f\left(\frac{pa+qb}{p+q}\right)\leqq\frac{1}{2x}\int_{A-x}^{A+x} f(t)\,\mathrm{d}t\leqq\frac{pf(a)+qf(b)}{p+q} \tag{18}$$

hold where $a_1\leqq a\leqq b\leqq b_1$, $p>0$, $q>0$, x satisfies condition (15) *and A is given by* (8).

In the special case when $p=q=1$, $x=\frac{b-a}{2}$, conditions (15) are fulfilled, so that from (18) inequalities (1) follows.

We shall now prove that the function

$$x\mapsto F(x)=\frac{1}{2x}\int_{A-x}^{A+x} f(t)\,\mathrm{d}t \tag{19}$$

is convex for $\max(a-A,\ A-b)\leqq x\leqq\min(A-a,\ b-A)$ if function f is convex and twice differentiable on the segment $[a, b]$. Under these conditions if we introduce the notation $G(x)=\int_A^x f(t)\,\mathrm{d}t$, we have $G'''(x)=f''(x)\geqq 0$ $(x\in[a, b])$.

Therefrom it follows that function G is convex of order 2 on $[a, b]$, i. e. for function G the inequality

$$(20)\qquad \frac{G(u_1)}{(u_1-u_2)(u_1-u_3)(u_1-u_4)}+\frac{G(u_2)}{(u_2-u_1)(u_2-u_3)(u_2-u_4)}+\frac{G(u_3)}{(u_3-u_1)(u_3-u_2)(u_3-u_4)}+\frac{G(u_4)}{(u_4-u_1)(u_4-u_2)(u_4-u_3)}\geqq 0,$$

is valid where $a\leqq u_i\leqq b$ $(i=1, 2, 3, 4)$. Putting in (20) primarily $u_i=A+x_i$ $(i=1, 2, 3)$, $u_4=A$ and $u_i=A-x_i$ $(i=1, 2, 3)$, $u_4=A$ and adding the inequalities obtained in such a way, we get

$$(21)\qquad \frac{\frac{1}{x_1}\int\limits_{A-x_1}^{A+x_1} f(t)\,dt}{(x_1-x_2)(x_1-x_3)}+\frac{\frac{1}{x_2}\int\limits_{A-x_2}^{A+x_2} f(t)\,dt}{(x_2-x_1)(x_2-x_3)}+\frac{\frac{1}{x_3}\int\limits_{A-x_3}^{A+x_3} f(t)\,dt}{(x_3-x_1)(x_3-x_2)}\geqq 0.$$

Inequality (20) is valid for all u_i such that $a\leqq u_i\leqq b$, wherefrom it follows that inequality (21) is valid if the conditions $a\leqq A+x_i\leqq b$ and $a\leqq A-x_i\leqq b$ are satisfied, i. e. if x_i's are chosen such that

$$\max(a-A,\ A-b)\leqq x_i\leqq \min(A-a,\ b-A).$$

In virtue of that, function F defined by (19) is convex for all x such that

$$(22)\qquad \max(a-A,\ A-b)\leqq x\leqq \min(A-a,\ b-A).$$

Therefrom it follows that function F reaches its minimal value when $x\to 0$. This minimal value is given by

$$\min F(x)=\lim_{x\to 0}\frac{1}{2x}\int\limits_{A-x}^{A+x} f(t)\,dt=f(A)=f\left(\frac{pa+qb}{p+q}\right).$$

Apart from condition (22) the variable x must also satisfy condition (15), so that the maximal value of function F is given by

$$\max F(x)=\min\left\{F\big(\min(A-a,\ b-A)\big),\ F\left(\frac{b-a}{p+q}\sqrt{3pq}\right)\right\}.$$

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