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467. ON N-FUNCTION AND ITS DERIVATIVE*

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The chief aim of this paper is to prove the necessary and sufficient condition for two functions to be mutually right — inverse. The same paper contains some inequalities for N-functions and some problems connected with them.

1. For a function $p(t)$ defined for $0 \leqq t \leqq +\infty$, we shall say that it belongs to the class P, if $p(t)$ satisfies

1° $p(0)=0$,

2° $p(t)>0$ for all $t>0$,

3° $p(t)$ is non-decreasing for all $t \geqq 0$,

4° $p(t)$ is continuous from the right for all $t \geqq 0$,

5° $p(+\infty)=\lim\limits_{t \to +\infty} p(t)=+\infty$.

It is known [1] that for every function $p(t) \in P$, the so called right inverse function $q(s)$ could be defined by

$$q(s)=\sup_{p(t) \leqq s} t. \tag{1}$$

It is easily proved that $q(s) \in P$, too (see [1]). On the other hand it is known [1] that for the functions $p(t)$ and $q(s)$, defined as above inequalities

$$p\big(q(s)\big) \geqq s,\ q\big(p(t)\big) \geqq t,\ p(q(s)-a) \leqq s,\ q(p(t)-a) \leqq t \tag{2}$$

hold, where $t, s \geqq 0$ and $a>0$ and naturally $p(t)-a \geqq 0$ and $q(s)-a \geqq 0$.

For functions $M(u)$ and $N(v)$ defined by

$$M(u)=\int_0^{|u|} p(t)\,\mathrm{d}t,\quad N(v)=\int_0^{|v|} q(s)\,\mathrm{d}s \tag{3}$$

it is said (see [1]) that they are complementary N-functions. There is a lot o proofs stating that YOUNG's inequality

$$uv \leqq M(u)+N(v) \tag{4}$$

is valid for all $u, v \in R$.

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If $u \geqq 0$ and $v \geqq 0$ then from (3) we have $M(u) \leqq up(u)$ and analogously $N(v) \leqq vq(v)$, which according to (4) means that inequality

$$uv \leqq up(u) + vq(v) \tag{5}$$

holds for all $u \geqq 0$, $v \geqq 0$ and $p(t)$, $q(s) \in P$ where functions $p(t)$ and $q(s)$ are inverse in the sense of (1). Inequality (5) is proved in **[3]** starting from inequality (4). However in **[3]** it was assumed that $p(t)$ was a continuous and a strictly increasing function, and that $q(s)$ is the ordinary inverse function of the function $p(t)$. It was noticed in [3] that inequality (5) is weaker but more effective than inequality (4).

A. C. ZAANEN considered in **[5]** functions $p_1(t)$, defined for all $t \geqq 0$ such that $p_1(t)$ is nondecreasing for $t \geqq 0$, $p_1(0) = 0$, $p_1(t)$ is left continuous for all $t \geqq 0$. The inverse function of $p_1(t)$ could be defined similarly as in (1). ZAANEN has proved that if $t, s \geqq 0$, then from $s < p_1(t)$ it follows $t > q_1(s)$ and from $s > p_1(t)$ it follows that $t \leqq q_1(s)$ (see lemma 1 in **[5]**). This result of ZAANEN could be formulated in the form of the following Lemma.

Lemma 1. *If functions $p_1(t)$ and $q_1(s)$ are defined as above, and if they are inverse in the aforementioned sense, then the inequality*

$$p_1(t)\, q_1(s) + ts \leqq tp_1(t) + sq_1(s), \tag{6}$$

holds for all $t \geqq 0$, $s \geqq 0$. Equality in (6) *holds if and only if $s = p_1(t)$ or $t = q_1(s)$.*

It is easy to see that inequality (6) holds even if $p(t)$ is substituted for $p_1(t)$ and if $q(s)$ is substituted for $q_1(s)$, where functions p and q are from the class P and q is defined by (1). Equality then holds if and only if $t = q(s)$ or $s = p(t)$.

Inequality (6) for the inverse functions $p(t)$, $q(s) \in P$ is sharper than inequality (5) because $p(t)\, q(s) \geqq 0$ for all $t, s \geqq 0$.

2. There is a certain number of papers in literature regarding so called inverse inequalities, for example, HÖLDER's and YOUNG's inequalities. In connection with that see for example **[2, 3, 4]**.

In the present paper we shall prove a lemma regarding the inversion of inequality (6) obtained in ZAANEN's paper **[5]**.

Lemma 2. *Let $p(t) \in P$. Then, if for any function $q(s)$, defined for all $s \geqq 0$ such that $q(0) = 0$, inequality*

$$p(t)\, q(s) + ts \leqq tp(t) + sq(s) \tag{7}$$

holds for all $t, s \geqq 0$, then

$$q(s) = \sup_{p(t) \leqq s} t,$$

i.e. $q(s)$ is then the right inverse function of $p(t)$.

Proof. Since $p(t) \in P$, there is a right inverse function $q^*(s)$ of the function $p(t)$ defined by (1) and also $q^*(s) \in P$. Therefore, inequality (2) holds for functions $p(t)$ and $q^*(s)$. Since inequality (7) is equivalent to

$$(p(t) - s)(q(s) - t) \leqq 0, \tag{8}$$

putting $t=q^*(s)\geqq 0\,(s\geqq 0)$ in (8), we get

$$(9)\qquad \big(p\big(q^*(s)\big)-s\big)\big(q(s)-q^*(s)\big)\leqq 0.$$

Since, from (2) follows $p\big(q^*(s)\big)\geqq s$ for $s\geqq 0$, from (9) we get

$$(10)\qquad q(s)\leqq q^*(s)\quad \text{(for all } s\geqq 0).$$

On the other hand, if $s>0$, let us choose any $h>0$ so that $q^*(s)-h\geqq 0$. Such a choice of h is possible, because $q^*(s)>0$ holds if $q^*(s)\in P$ and if $s>0$. Hence, if we take $0<h_1\leqq h$, we get that $q^*(s)-h_1\geqq 0$. Introducing the substitution $t=q^*(s)-h_1$ into (8) we find

$$(11)\qquad \big(p(q^*(s)-h_1)-s\big)\big(q(s)-q^*(s)+h_1\big)\leqq 0.$$

For this choice of s and h_1, according to (2) we get $p(q^*(s)-h_1)\leqq s$, and on the basis of (11) we get

$$q(s)-q^*(s)+h_1\geqq 0.$$

Since the last inequality holds for all h_1 such that $0<h_1\leqq h$, letting $h_1\to 0+$, we obtain

$$(12)\qquad q(s)\geqq q^*(s)\qquad \text{(for all } s>0).$$

From inequalities (10) and (12) we get

$$(13)\qquad q^*(s)=q(s),$$

for all $s>0$. Thus, since the previous equality (13) holds for all $s>0$, on the basis of right continuity of the function $q^*(s)\in P$ we get

$$\lim_{s\to 0+} q^*(s)=\lim_{s\to 0+} q(s)=0.$$

Since $q(0)=0$, $q(s)$ is right continuous at the point $s=0$, which means that equality (13) is valid for all $s\geqq 0$. Hence the lemma is proved.

On the basis of the proof of Lemma 2, it follows immediately that the assumption $q(0)=0$ can be replaced by the supposition that $q(s)$ is right continuous at the point $s=0$.

From Lemma 1 and Lemma 2, we directly get the following theorem.

Theorem 1. *Let $p(t)\in P$. Let the function $q(s)$ be defined for all $s\geqq 0$ and let $q(0)=0$ (or let $q(s)$ be right continuous for $s=0$).*

Then the necessary and sufficient condition for the function $q(s)$ to be the right inverse function of the function $p(t)$ (or that $q(s)$ satisfies (1)), is that inequality (7) holds for all $t\geqq 0$ and $s\geqq 0$.

It is clear that $q(s)$ is a unique function for which the conditions of the above theorem are fulfilled. Furthermore, it is logical to ask whether the assumption $q(0)=0$, i.e. the assumption that $q(s)$ is right continuous for $s=0$ could be dropped from Theorem 1. A negative reply to the above question is provided by the following example.

EXAMPLE. Function $p(t)=2t\,(t\geqq 0)$ belongs to the class P. The right inverse function of this function is $q^*(s)=\frac{1}{2}s\,(s\geqq 0)$. On the other hand function

$$q(s)=\frac{1}{2}s\,(s>0), \quad q(s)=-1\,(s=0),$$

satisfies inequality (7) for all $t\geqq 0$ and $s\geqq 0$, which can be directly verified, but neither $q(s)\in P$ nor $q(s)$ is the right inverse function of the function $p(t)$.

In [3] page 124, theorems 170 and 171, on the basis of inequality (5) it is concluded that if the series $\sum_{k=1}^{+\infty} a_k p(a_k)$ and $\sum_{k=1}^{+\infty} b_k q(b_k)$, where $a_k\geqq 0$ and $b_k\geqq 0$ $(k=1, 2, \ldots)$, $p(t)\in P$ and $q(s)$ is defined by (1), are convergent, then the series $\sum_{k=1}^{+\infty} a_k b_k$ converges. On the other hand it was shown that the series $\sum_{k=1}^{+\infty} a_k p(a_k)$ can diverge in those cases when the series $\sum_{k=1}^{+\infty} a_k b_k$ is convergent for all convergent series $\sum_{k=1}^{+\infty} b_k q(b_k)$. However, using inequality (7) we can prove the following theorem.

Theorem 2. *Let* $a_k\geqq 0$, $b_k\geqq 0$ $(k=1, 2, \ldots)$ *and let* $p(t)\in P$, $q(s)$ *being defined by* (1). *If the series*

$$\sum_{k=1}^{+\infty} a_k p(a_k) \quad \text{and} \quad \sum_{k=1}^{+\infty} b_k q(b_k)$$

converge then the series

$$\sum_{k=1}^{+\infty} \big(a_k b_k + p(a_k)\, q(b_k)\big) \tag{14}$$

also converges and furthermore the series

$$\sum_{k=1}^{+\infty} a_k b_k \quad \text{and} \quad \sum_{k=1}^{+\infty} p(a_k)\, q(b_k)$$

are convergent.

The validity of the following two statements remains to be investigated:

(a) If the series (14) converges for every convergent series $\sum_{k=1}^{+\infty} a_k p(a_k)$ then the series $\sum_{k=1}^{+\infty} b_k q(b_k)$ also converges.

(b) Is it possible to derive YOUNG's inequality from the inequality (7)?

Let $p(t)$ and $q(s)$ be the inverse functions of class P. Integrating (7) with respect to t, from 0 to $u>0$, we have

$$q(s)\int_0^u p(t)\,dt + s\frac{u^2}{2} \leqq \int_0^u t p(t)\,dt + usq(s).$$

In the same manner, integrating the last inequality with respect to s, from 0 to $v>0$, we get

$$M(u)\,N(v)+\frac{u^2v^2}{4}\leqq v\int_0^u tp(t)\,dt+u\int_0^v sq(s)\,ds.$$

Since partial integration yields

$$\int_0^u tp(t)\,dt=uM(u)-\int_0^u M(t)\,dt,\quad \int_0^v sq(s)\,ds=vN(v)-\int_0^v N(s)\,ds,$$

on the basis of the previously derived inequality we get:

Theorem 3. *For any real numbers u and v and any complementary N-functions $M(u)$ and $N(v)$ we have*

$$M(u)\,N(v)+|v|\int_0^{|u|} M(t)\,dt+|u|\int_0^{|v|} N(s)\,ds+\frac{u^2v^2}{4}\leqq|uv|\,\big(M(u)+N(v)\big).$$

If $u\geqq 0$ and $v\geqq 0$, then inequality

$$M(u)\,N(v)\leqq uv\,\big(M(u)+N(v)\big)$$

follows directly from the above inequality. Introducing the substitutions $a=M(u)$ and $b=N(v)$ where $a>0$ and $b>0$, we get $ab\leqq(a+b)\,M^{-1}(a)\,N^{-1}(b)$, i.e. the inequality

$$\frac{ab}{a+b}\leqq M^{-1}(a)\,N^{-1}(b)$$

holds. On the other hand from YOUNG's inequality, using the same substitutions we find that $M^{-1}(a)\,N^{-1}(b)\leqq a+b$. Hence we obtain:

Theorem 4. *For any complementary N-functions $M(u)$ and $N(v)$ and any positive numbers a and b we have*

$$\frac{ab}{a+b}\leqq M^{-1}(a)\,N^{-1}(b)\leqq a+b.$$

This inequality is a generalisation of an inequality obtained in **[1]** (page 13, inequality 2.10.).

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