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465. ON AN INEQUALITY FOR CONVEX FUNCTIONS OF ORDER k*

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Let $F_1, F_2, \dots, F_n$ be integrable nonnegative and monotone functions in the same sense on $[a, b]$ and let P be a positive and integrable function on $[a, b]$. Then the following inequality of ČEBYŠEV holds

$$(1)\qquad \Big(\int_a^b P(x)\,\mathrm{d}x\Big)^{n-1}\Big(\int_a^b P(x)\,F_1(x)\cdots F_n(x)\,\mathrm{d}x\Big) \geqq \Big(\int_a^b P(x)\,F_1(x)\,\mathrm{d}x\Big)\cdots\Big(\int_a^b P(x)\,F_n(x)\,\mathrm{d}x\Big).$$

Equality in (1) holds if and only if $F_i(x)=C_i\,(C_i=\text{const})$ $(i=1, 2, \dots, n)$.

It is known that if some additional restrictions are introduced for function F_i, inequality (1) could be improved. Namely, P. M. VASIĆ [1] has proved the following:

Theorem 1. *If $f_1, \dots, f_n$ are integrable and convex functions on $[a, b]$ such that $f_k(x)\geqq 0$ for $x\in[a, b]$ and if $f_k(a)=0$ $(k=1, \dots, n)$ and $p(x)$ is a positive and integrable function on $[a, b]$ we have*

$$(2)\qquad \Big(\int_a^b p(x)\,\mathrm{d}x\Big)^{n-1}\Big(\int_a^b p(x)\,f_1(x)\cdots f_n(x)\,\mathrm{d}x\Big) \geqq M\,\Big(\int_a^b p(x)\,f_1(x)\,\mathrm{d}x\Big)\cdots\Big(\int_a^b p(x)\,f_n(x)\,\mathrm{d}x\Big)$$

where

$$(3)\qquad M=\frac{\Big(\int_a^b p(x)\,(x-a)^n\,\mathrm{d}x\Big)\Big(\int_a^b p(x)\,\mathrm{d}x\Big)^{n-1}}{\Big(\int_a^b p(x)\,(x-a)\,\mathrm{d}x\Big)^n}.$$

* Presented December 10, 1973 by P. M. VASIĆ.

Equality in (2) *holds if and only if* $f_k(x)=c_k(x-a)$ $(k=1, \dots, n)$.

Since, on the basis of (1), $M \geqq 1$, we see that (2) is a sharper inequality than (1).

Putting $p(x)=1$, $a=0$, $b=1$, we have $M=\frac{2^n}{n+1}$, and (2) yields inequality of B. J. ANDERSSON ([**2**], see also [3], p. 306)

$$\int_0^1 f_1(x_1)\cdots f_n(x)\,dx \geqq \frac{2^n}{n+1}\Big(\int_0^1 f_1(x)\,dx\Big)\cdots\Big(\int_0^1 f_n(x)\,dx\Big). \tag{4}$$

We shall show that inequality (2) can be improved under certain conditions. Primarily, we shall quote the following result, which may be proved by application of mathematical induction.

Theorem 2. *Let f be a nonnegative function on* $[a, b]$, *satisfying conditions:*

1° *f is a convex function of order k,*

2° $$\lim_{x\to a}\frac{f(x)}{(x-a)^r}=0 \qquad (r=0, 1, \dots, k-2),$$

then function $x \mapsto \frac{f(x)}{(x-a)^{k-1}}$ *is nondecreasing.*

Let us put in (1) $n=2$ and

$$P(x)=p(x)(x-a)^{k-1},\quad F_1(x)=((x-a)^{k-1})^{n-1},\quad F_2(x)=\frac{f_1(x)\dots f_n(x)}{((x-a)^{k-1})^n}$$

where functions $f_1, \dots, f_n$ satisfy conditions of theorem 2. Then we have

$$\Big(\int_a^b p(x)(x-a)^{k-1}\,dx\Big)\Big(\int_a^b p(x) f_1(x)\cdots f_n(x)\,dx\Big) \tag{5}$$

$$\geqq\Big(\int_a^b p(x)((x-a)^{k-1})^n\,dx\Big)\Big(\int_a^b p(x)\frac{f_1(x)\cdots f_n(x)}{((x-a)^{k-1})^{n-1}}\,dx\Big).$$

Putting in (1)

$$P(x)=p(x)(x-a)^{k-1},\ F_i(x)=\frac{f_i(x)}{(x-a)^{k-1}} \qquad (i=1, 2, \dots, n)$$

we get

$$\Big(\int_a^b p(x)(x-a)^{k-1}\,dx\Big)^{n-1}\Big(\int_a^b p(x)\frac{f_1(x)\cdots f_n(x)}{((x-a)^{k-1})^{n-1}}\,dx\Big) \tag{6}$$

$$\geqq\Big(\int_a^b p(x) f_1(x)\,dx\Big)\cdots\Big(\int_a^b p(x) f_n(x)\,dx\Big).$$

By combining inequalities (5) and (6) we get

$$(7)\qquad \Big(\int_a^b p(x)\,dx\Big)^{n-1}\Big(\int_a^b p(x) f_1(x)\cdots f_n(x)\,dx\Big) \geqq M_k \Big(\int_a^b p(x) f_1(x)\,dx\Big)\cdots\Big(\int_a^b p(x) f_n(x)\,dx\Big)$$

where

$$(8)\qquad M_k=\frac{\Big(\int_a^b p(x)\big((x-a)^{k-1}\big)^n\,dx\Big)\Big(\int_a^b p(x)\,dx\Big)^{n-1}}{\Big(\int_a^b p(x)(x-a)^{k-1}\,dx\Big)^n}.$$

Equality in (7) holds if and only if $f_i(x)=c_i(x-a)^{k-1}$ $(i=1, 2, \dots, n)$. Putting $k=2$ we have $M_2=M$, and (7) becomes (2). Therefore we have:

Theorem 3. *If $f_1, \dots, f_n$ are integrable nonnegative and convex functions of order k on $[a, b]$, such that* $\lim\limits_{x\to a}\dfrac{f(x)}{(x-a)^r}=0$ $(r=0, 1, \dots, k-2)$, *then inequality (7) holds; M_k being given by* (8).

We shall prove that $M_k \geqq M_{k-1}$, i.e. that by increasing the order of convexity, much better inequalities are obtained.

Introduce in (1)

$$P(x)=p(x)(x-a)^{k-2},\quad F_1(x)=\big((x-a)^{k-2}\big)^{n-1},\quad F_i(x)=x-a\quad (i=2, 3, \dots, n+1),$$

we obtain

$$(9)\qquad \Big(\int_a^b p(x)(x-a)^{k-2}\,dx\Big)^n\Big(\int_a^b p(x)\big((x-a)^{k-1}\big)^n\,dx\Big) \geqq \Big(\int_a^b p(x)\big((x-a)^{k-2}\big)^n\,dx\Big)\Big(\int_a^b p(x)(x-a)^{k-1}\,dx\Big)^n$$

or

$$(10)\qquad \frac{\int_a^b p(x)\big((x-a)^{k-1}\big)^n\,dx}{\Big(\int_a^b p(x)(x-a)^{k-1}\,dx\Big)^n} \geqq \frac{\Big(\int_a^b p(x)(x-a)^{k-2}\Big)^n\,dx}{\Big(\int_a^b p(x)(x-a)^{k-2}\,dx\Big)^n}.$$

Since on the basis of (10), $M_k \geqq M_{k-1}$, we see that (7) is a sharper inequality than (2). The above sharper inequality is obtained by adding the order k convexity condition.

REFERENCES

1. P. M. Vasić: *On an inequality for convex functions.* These Publications **№ 381—№ 409** (1972), 67—70.

2. B. J. Andersson: *An inequality for convex functions.* Nordisk Mat. Tidskr. **6** (1958), 25—26.

3. D. S. Mitrinović: *Analytic Inequalities.* Berlin-Heidelberg-New York, 1970.

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