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463. RECIPROCAL TRANSFORMATIONS OF POTENTIAL AND ČEBYŠEV'S DEVELOPMENTS*

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Economization of polynomials has a great significance in computer calculations, since a lot of functions in a given interval could be approximated by polynomials. Owing to the computer techniques, it is of use to apply the economization for calculations of elementary and higher transcendental functions, in frequently used programs. In that way the number of multiplication and subtraction operations could be considerably reduced, which increases the speed and accuracy of the program.

Some computer Companies in their program package supply a program for economization of the polynomials, which, by ČEBYŠEV's development of the highest degree term of the polynomial, modifies the other polynomial terms, so forming the approximative polynomial of a lower order. The process of lowering the polynomial order and modifying its members goes on until the approximation error exceeds the permitted value (see for example [**1**], pp. 178—181). For the slowly convergent series, the number of iterations is extremely high, and such a procedure accumulates a great error, so that the economization must be carried out using a computer of greater accuracy than the one to which the results will be applied.

A program having general advantages with respect to the above mentioned, was proposed in [**2**], pp. 46—59, enabling primarily the transformation of the original polynomial into ČEBYŠEV's development, and then the transformation of the shortened ČEBYŠEV's development into an approximative polynomial. These two transformations are not simultaneous, but entirely independent, so that each can be separately used. Separating transformations, the accumulation of the error occurring in slowly-convergent potential developments, which mostly need the economization, can be prevented. However, by the program given in [**2**], only the polynomial up to the ninth order can be approximated, because the coefficients for these two transformations are supplied as the elements of two matrices of order 10×10.

Program TCNP for the transformation of ČEBYŠEV's development into potential one is given in [**1**], p. 200—201, based on the direct application of the formulae

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x).$$

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Transformation of potential development into ČEBYŠEV's is not given in [1], which is, perhaps, a consequence of the fact that certain distinguished books do not supply the exact ČEBYŠEV's development for x^m.

This paper presents the correction of the formula for the x^m development by ČEBYŠEV's polynomials. The algorithms generating coefficients for the transformation of potential into ČEBYŠEV's development and vice versa are proposed, in such a way that the order of the polynomials which are to be economized may be selected. Instead of an $n \times n$ matrix, an nth order vector is used, so that even the memory of a mini-computer is sufficient.

The given algorithm does not accumulate the error, so that the economization of slowly-convergent potential developments can be successfully carried out.

1. We shall present at first the well-known procedure for economization of potential developments of functions. Let the function $z \mapsto f(z)$ in segment $-r \leqq z \leqq r$ be approximately represented by polynomial

$$p_{n-1}(z) = \sum_{k=1}^{n} a_k z^{k-1} \qquad (|z| \leqq r). \tag{1}$$

By a linear transformation of coordinates, $z = rx$, polynomial p_{n-1} is transformed into

$$p_{n-1}(z) = \sum_{k=1}^{n} b_k x^{k-1} \qquad (|x| \leqq 1). \tag{2}$$

Thereafter, using transformation of x^{k-1} into the development by ČEBYŠEV's polynomials

$$x^i = \sum_{j=0}^{i} c_{ij} T_j, \tag{3}$$

ČEBYŠEV's development of the given polynomial p_{n-1} is obtained

$$p_{n-1}(z) = \sum_{j=1}^{n} d_j T_{j-1}. \tag{4}$$

Since $|T_j| \leqq 1$, the following majorization is obtained

$$\left| \sum_{k=1}^{n} d_k T_{k-1} - \sum_{k=1}^{m} d_k T_{k-1} \right| \leqq \sum_{k=1}^{n} |d_k| - \sum_{k=1}^{m} |d_k| \qquad (m \leqq n).$$

If the modulus of the permitted error is ε, the necessary and sufficient order m of the polynomial

$$q_{m-1}(z) = \sum_{j=1}^{m} d_j T_{j-1} \tag{5}$$

created by the economization of the polynomial p_{n-1} is obtainable from the conditions

$$\sum_{k=1}^{n} |d_k| - \sum_{k=1}^{m} |d_k| \leqq \varepsilon \quad \text{and} \quad \sum_{k=m}^{n} |d_k| > \varepsilon.$$

The value of the polynomial $q_{m-1}(z)$ can be directly obtained on the basis of (5), for example in the manner given in [**3**], p. 427, or [**1**], p. 199. Applying inverse transformations, the polynomial in z is again obtained. On the basis of (5) and of the development

$$T_i = \sum_{j=1}^{i} e_{ij} x^j, \tag{6}$$

we obtain

$$q_{m-1}(z) = \sum_{i=1}^{m} f_i x^{i-1} \qquad (|x| \leqq 1) \tag{7}$$

and therefrom, by a linear transformation $x = \dfrac{z}{r}$,

$$q_{m-1}(z) = \sum_{k=1}^{m} g_k z^{k-1} \qquad (|z| \leqq r). \tag{8}$$

If $\varepsilon < |d_n|$, then $m = n$, so that $g_k = a_k$ $(k = 1, \ldots, n)$, which enables us to check the algorithm and to follow numerical errors.

2. Let us adopt the function $G(t) = \dfrac{1-tx}{1-2tx+t^2}$ for the generatrix of Čebyšev's polynomials, so that Čebyšev's polynomials for $n = 0, 1, 2, \ldots$ can be introduced by

$$\frac{1-tx}{1-2tx+t^2} = \sum_{n=0}^{+\infty} T_n(x) t^n, \tag{9}$$

where t is the complex variable and $-1 \leqq x \leqq 1$. Using the substitution $x = \cos\theta$, for $|t| < 1$, the generating function $G(t)$ becomes

$$G(t) = \frac{1-tx}{1-2tx+t^2} = \frac{1}{2}\left(\frac{1}{1-te^{i\theta}} + \frac{1}{1-te^{-i\theta}}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{+\infty} t^n e^{in\theta} + \sum_{n=0}^{+\infty} t^n e^{-in\theta}\right) = \sum_{n=0}^{+\infty} t^n \cos n\theta,$$

wherefrom using (9), it follows

$$T_n(\cos\theta) = \cos n\theta. \tag{10}$$

By differentiating (10) it is directly proved that Čebyšev's polynomials satisfy the differential equation

$$(1-x^2) y'' - xy' + n^2 y = 0. \tag{11}$$

Starting from the identity

$$2\cos\theta \cos n\theta = \cos(n+1)\theta + \cos(n-1)\theta,$$

using the substitution $\cos\theta = x$ for $n = 1, 2, \ldots$, we obtain

$$2x T_n = T_{n+1} + T_{n-1}; \tag{12}$$

(compare [**4**], pp. 75, 78 and 80). Though the known formulas (10), (11) and (12) suggest the relation $T_{-n}(x) = T_n(x)$, we shall consider Čebyšev's polynomials $x \mapsto T_n(x)$ only for $n = 0, 1, 2, \ldots$

3. In [**4**], p. 18, polynomials $x \mapsto T_n(x)$ for $n=0\,(1)\,10$ are given, as well as the general formula

$$T_0(x)=1,\ \ T_n(x)=\frac{n}{2}\sum_{k=0}^{[n/2]}\frac{(-1)^k(n-k-1)!}{k!\,(n-2k)!}(2x)^{n-2k} \qquad (n=1,2,\ldots). \tag{13}$$

Also in [**4**], p. 89, the developments of the functions x^m with respect to T_0, $T_1, \ldots, T_m$ for $m=0\,(1)\,12$ are given. In [**5**], p. 298, it is claimed that formula

$$x^m=2^{1-m}\sum_{k=0}^{[m/2]}\binom{m}{k}T_{m-2k} \qquad (m>0)$$

is true.

Among other things we shall prove that this important formula for x^m should read

$$x^m=2^{1-m}\sum_{k=0}^{[m/2]}\frac{\binom{m}{k}T_{m-2k}}{1+\delta_{m,\,2k}} \qquad (m=0,\ 1,\ 2,\ \ldots), \tag{14}$$

where δ_{ij} is Kronecker's symbol: $\delta_{ij}=1$ $(i=j)$, $\delta_{ij}=0$ $(i\neq j)$.

Let us introduce the notation $\delta\equiv\delta_{n/2,\,[n/2]}$. Formula (14) will be proved by mathematical induction. For $m=0$ and $m=1$, formula (14) is true, because $T_0(x)=1$ and $T_1(x)=x$. Let us suppose that (14) is true for $m=n>1$, i. e., that

$$x^n=2^{-n}\sum_{k=0}^{[n/2]-1}\binom{n}{k}2T_{n-2k}+2^{-n}\binom{n}{[n/2]}(\delta T_0+(1-\delta)\,2T_1), \tag{15}$$

is valid. Multiplying (15) by x and using the recurrent relation for Čebyšev's polynomials (12) we get

$$x^{n+1}=2^{-n}\left(\sum_{k=0}^{[n/2]-1}\binom{n}{k}T_{n+1-2k}+\sum_{k=1}^{[n/2]}\binom{n}{k-1}T_{n+1-2k}+\binom{n}{[n/2]}\big(\delta T_1+(1-\delta)(T_2+T_0)\big)\right).$$

Using the well-known formula $\binom{i}{j}+\binom{i}{j+1}=\binom{i+1}{j+1}$, x^{n+1} becomes

$$x^{n+1}=2^{-n}\left(T_{n+1}+\sum_{k=1}^{[n/2]-1}\binom{n+1}{k}T_{n+1-2k}+\binom{n+1}{[n/2]}\big((1-\delta)\,T_2+\delta T_1\big)+\binom{n}{[n/2]}(1-\delta)T_0\right),$$

and hence using

$$\binom{n+1}{0}=1 \quad\text{and}\quad \binom{n}{[n/2]}(1-\delta)=\binom{n+1}{[(n+1)/2]}\frac{1-\delta}{2},$$

we get

$$x^{n+1}=2^{-n}\sum_{k=0}^{[(n+1)/2]}\frac{\binom{n+1}{k}T_{n+1-2k}}{1+\delta_{n+1,\,2k}}$$

which is in fact formula (14) for $m=n+1$. The proof of (14) by mathematical induction is completed.

Using certain relations for ČEBYŠEV's polynomials $x \mapsto T_i(x)$ of the negative index i and using equality

$$x^m T_n(x) = 2^{-m} \sum_{k=0}^{m} \binom{m}{k} T_{m-2k+n}(x)$$

for $n=0$, see [4], p 80, it is possible to prove (14) in an easier way.

4. Computer realization of the given algorithm requires further specifications and transformations, so that the errors, always occurring in work with digital computer are reduced to the smallest extent.

The transfer from (1) to (2) is carried out by

$$b_i = r^{i-1} a_i \qquad (i=1, \ldots, n).$$

Upon short calculations, starting from (3) and (11), we get

$$c_{ij} = 2^{2-i} \delta_{2k,\,2[k]} \binom{i-1}{k} \Big/ (1+\delta_{1,j}) \qquad \left(k = \frac{i+j-2}{2} \geqq 0\right), \tag{16}$$

so that the transfer from development (2) to development (4) could be made by the formula (16) and

$$d_j = \sum_{i=j}^{n} b_i c_{ij} \qquad (j=1, \ldots, n).$$

Upon calculation of the approximation error, i. e., upon the determination of the number of summations m of the polynomial (5), obtained by economization, it is necessary to ensure the transfer from (5) to (7). For that we use the result

$$e_{ij} = (-1)^{1+k-j} 2^{j-2} \delta_{2k,\,2[k]} \binom{k}{j-1} \frac{i-1}{k} \left(k = \frac{i+j-2}{2} > 0\right),\ e_{11} = 1. \tag{17}$$

which is derived from (6) and (13), as well as

$$f_j = \sum_{i=j}^{m} d_i e_{ij} \qquad (j=1, \ldots, m).$$

Finally, using formula (7) and

$$g_i = r^{2-j} f_j \qquad (j=1, \ldots, m)$$

we get development (8).

Owing to the limitations of the computer memory the vector of dimension n should be used instead of matrices c_{ij} and e_{ij} of dimension $n \times n$. It will enable the economization of even very slowly convergent developments. That is the reason why it is indispensable to use the given formulas (14) and (16) for finding the algorithms for generating these coefficients. In such a way the memory cells where the used coefficients were located are in next step used for storage of new coefficients. This means that an entirely general algorithm is created which may be realized on every computer. From (16) it follows

$$c_{11} = 1, \quad c_{i1} = \frac{i-2}{i-1} c_{i-2,1} \qquad \left(i = 3, 5, \ldots, 2\left[\frac{n}{2}\right] - 1\right),$$

$$c_{ij} = \frac{i}{1+j} (1+\delta_{2j})\, c_{i-1,\,j-1} \qquad \left(j>1;\ i=j,\ j+2, \ldots, j+2\left[\frac{n-j}{2}\right]\right).$$

From (17) it follows

$$e_{i1}=1, \quad e_{i,0}=-e_{i-2,0}\left(i=3,5,\ldots,2\left[\frac{m}{2}\right]-1\right),$$

$$e_{ij}=2\,e_{i-1j-1}-e_{i-2,j}\left(j>1;\ i=j,\ j+2,\ \ldots,j+2\left[\frac{m-j}{2}\right]\right).$$

In order to avoid excessive calculations and decreases in accuracy of calculations it is necessary to substitute coefficients b_i^*, c_{ij}^*, e_{ij}^*, $f_{i,j}^*$ for b_i, c_{ij}, e_{ij}, f_i in the following manner:

$$b_i^*=2^{1-i}\,b_i, \quad c_{ij}^*=2^{i-1}\,(1+\delta_{0j})\,c_{ij}, \quad e_{ij}^*=2^{1-j}\,e_{ij}, \quad f_j^*=2^{j-1}f_j.$$

5. Program PNTN transforms the potential development

$$a_1+a_2z+\cdots+a_nz^{n-1} \qquad (|z|\leqq r)$$

into Čebyšev's

$$d_1T_0(x)+d_2T_1(x)+\cdots+d_nT_{n-1}(x) \qquad (|x|\leqq 1). \tag{18}$$

W_m denotes the error occurring if the last term in Čebyšev's development (18) is $d_mT_{m-1}(x)$ $(1\leqq m\leqq n)$.

```
      SUBROUTINE PNTN (N,R,A,D,W)
      DIMENSION A(1),D(1),W(1)
      P=2.
      Q=R*0.5
      DO 10 I=1,N
      D(I)=A(I)*P
   10 P=P*Q
      J=1
      W(1)=1.
      DO 20 I=3,N,2
   20 W(I)=W(I-2)*(4*I-8)/(I-1)
      GO TO 50
   30 DO 40 I=J,N,2
   40 W(I)=W(I-1)*(I+I-2)/(I+J-2)
   50 Q=0.
      I=(N-J)/2*2+J
      GO TO 70
   60 Q=D(I)*W(I)+Q
      I=I-2
   70 IF(I-J) 80,80,60
   80 D(J)=D(J)+Q
      J=J+1
      IF(J-N) 30,90,90
   90 D(1)=D(1)*0.5
      W(J)=0.
      GO TO 130
  100 P=D(J)
      IF(P) 110,120,120
  110 P=-P
  120 W(I)=W(J)+P
      J=I
  130 I=J-1
      IF(I) 140,140,100
  140 RETURN
      END
```

```
      SUBROUTINE TNSM (N,M,R,D,S)
      DIMENSION D(1),S(1)
      P=2.
      T=R/P
      J=1
      S(1)=P
      DO 10 I=3,N,2
   10 S(I)=-S(I-2)
      GO TO 50
   20 S(J)=1.
      I=J
      GO TO 40
   30 S(K)=S(I+1)-S(I)
      I=K
   40 K=I+2
      IF(K-N) 30,30,50
   50 Q=0.
      I=(N-J)/2*2+J
      GO TO 70
   60 Q=D(I)*S(I)+Q
      I=I-2
   70 IF(I-M) 80,80,60
   80 S(J)=Q/P
      P=P*T
      J=J+1
      IF(J-M) 20,20,90
   90 RETURN
      END
```

Program TNSM transforms the part of ČEBYŠEV's development

$$d_{m+1} T_m(x) + d_{m+2} T_{m+1}(x) + \cdots + d_n T_{n-1}(x) \qquad (|x| \leqq 1)$$

that is not used into the correction of the starting development

$$(a_1 - g_1) + (a_2 - g_2) z + \cdots + (a_m - g_m) z^{m-1} \qquad (|z| < r).$$

* * *

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