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462. ON A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS*

Radomir P. Lučić

The topic of this paper is the following functional-differential equation

$$F_0\big(x, f(x), f'(x), \ldots, f^{(k)}(x)\big) = f\big(g(x)\big) \tag{1}$$

where f is an unknown function. In this article we shall prove the following theorem.

Theorem. *Let the following conditions be satisfied:*

1° *The function g maps the open set G in G, G being the subset of the set R of real numbers.*

2° *Function g has iterations such that*

$$g_1(x) = g(x), \ldots, g_r(x) = g\big(g_{r-1}(x)\big), \ldots, g_n(x) = x$$

for each $x \in G$, where n is the smallest natural number for which the last expression holds.

3° *Function g has derivatives up to, and inclusive of, the order $nk-k$ for each $x \in G$, $g'(x) \neq 0$ for each $x \in G$.*

4° *Function $F_0(x, u_1, u_2, \ldots, u_{k+1})$ is nk—k times differentiable of its arguments for each $x \in G$ and $u_r \in R$ $(r = 1, \ldots, k+1)$ and $\dfrac{\partial F_0}{\partial u_{k+1}} \neq 0$.*

5° *The unknown function f has derivatives up to, and inclusive of, the order nk on G.*

In this case there exists an ordinary differential equation of order nk such that each solution of equation (1) *is simultaneously the solution of this differential equation.*

Proof. If we differentiate equation (1) nk—k times, which is justifiable due to 3° and 4°, we obtain for $r = 0, 1, \ldots, nk$—k

$$F_r\big(x, f(x), f'(x), \ldots, f^{(k+r)}(x)\big) = f^{(r)}\big(g(x)\big) \tag{2}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}x}(F_{s-1}) = F_s \frac{\mathrm{d}g}{\mathrm{d}x} \quad (s = 1, \ldots, nk-k).$$

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We shall prove, primarily, that

$$f(g_m(x)) = H_m(x, f(x), f'(x), \dots, f^{(km)}(x)) \qquad (m \leq n) \tag{3}$$

where H_m is one of the functions of the mentioned arguments. Since according to (2),

$$f(g_1(x)) = F_0(x, f(x), f'(x), \dots, f^{(k)}(x))$$

proposition (3) is valid for $m = 1$.

Let proposition (3) be valid for $m\,(m < n)$, then substituting $g(x)$ for x in (3), having in view 1° and 2° we obtain

$$f(g_{m+1}(x)) = H_m(g(x), f(g(x)), f'(g(x)), \dots, f^{(km)}(g(x)))$$

i.e.

$$f(g_{m+1}(x)) = H_m(g(x), F_0, F_1, \dots, F_{km})$$

hence

$$f(g_{m+1}(x)) = H_{m+1}(x, f(x), f'(x), \dots, f^{(km+k)}(x))$$

and finally we conclude that (3) is valid for $m+1$. Thereby the validity of this statement is proved.

Putting $m = n$ in (3) and taking into account 2° and 5° we obtain the ordinary differential equation

$$f(x) = H_n(x, f(x), f'(x), \dots, f^{(nk)}(x)) \tag{4}$$

of order nk, which completes the proof.

Corollary. If the point $x_0 \in G$ exists such that $g(x_0) = x_0$ and if the conditions $1° - 5°$ are satisfied then the solution of equation (4) with initial conditions

$$f(x_0) = C_1, \dots, f^{(k-1)}(x_0) = C_k,$$

$$F_r(x_0, f(x_0), f'(x_0), f^{(k+r)}(x_0)) = f^{(r)}(x_0) \quad (r = 0, 1, \dots, nk-k)$$

is at the same time the solution of equation (1) with initial conditions.

$$f(x_0) = C_1, \dots, f^{(k-1)}(x_0) = C_k.$$

In **[1]** the case $k = 1$, $n = 2$, is treated, and in **[2]** a special linear functional-differential equation is observed where $n = 2$ and $k = m$.

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