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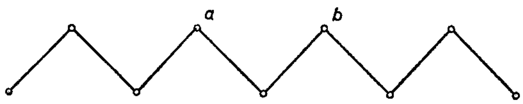
451. ENUMERATION OF A SPECIAL CLASS OF PERMUTATIONS BY RISES*

*Leonard Carlitz***

1. It is well known (see for example [**3**, pp 105—112]) that if $A(n)$ denotes the number of up-down permutations of $Z_n=\{1, 2, \ldots, n\}$, then $(A(0)=A(1)=1)$

$$\sum_{n=0}^{\infty} A(n)\frac{x^n}{n!}=\sec x+\tan x. \tag{1.1}$$

The writer has refined this result in the following way. Let $A(n, r)$ denote the number of up-down permutations of Z_n with r rises on the top line:

$A(n, r)$: 

A *rise* is a pair of consecutive elements a, b with $a<b$; also we agree to count a conventional rise on the left. For example

132546, 426153

have 3 and 2 rises, respectively. It has been proved [**1**] that

$$\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}\sum_r A(2n+1, r)x^r=\frac{U'(z)}{U(z)} \tag{1.2}$$

and

$$\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}\sum_r A(2n, r)x^r=1-x+\frac{x}{U(z)}, \tag{1.3}$$

where

$$A(0, r)=\delta_{0,r}, \quad A(1, r)=\delta_{0,r-1}$$

and

$$U(z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}\prod_{k=0}^{n-1}\left(1+2k(1-x)\right). \tag{1.4}$$

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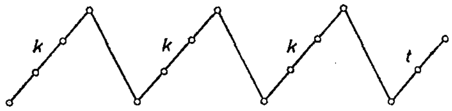
One can generalize up-down permutations in the following way. Let k, t be fixed integers, $k \geqq 2$, $t \geqq 0$ and consider permutations of Z_{kn+t} of the type

(1.5)



For brevity we may call permutations of this kind (k, t)-permutations.

Let $A_{k,t}(kn+t)$ denote the number of (k, t)-permutations of Z_{kn+t}. The writer has proved [2] that

$$\sum_{n=0}^{\infty} A_{k,0}(kn) \frac{z^{kn}}{(kn)!} = \frac{1}{\Phi_{k,0}(z)} \tag{1.6}$$

and

$$\sum_{n=0}^{\infty} A_{k,t}(kn+t) \frac{z^{kn+t}}{(kn+t)!} = \frac{\Phi_{k,t}(z)}{\Phi_{k,0}(z)} \qquad (t>0), \tag{1.7}$$

where

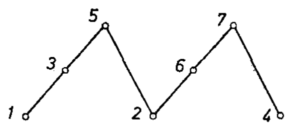
$$\Phi_{k,t}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{kn+t}}{(kn+t)!} \qquad (t \geqq 0). \tag{1.8}$$

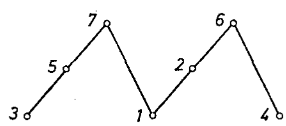
In the present paper we consider the refined problem of enumerating (k, t)-permutations of Z_{kn+t} with a given number of rises on the top line (see (1.5)). Let $A_{k,t}(kn+t, r)$ denote the number of (k, t)-permutations with r rises on the top line. Explicit formulas for the generating functions

$$F_{k,t}(z) = \sum_{n=0}^{\infty} \sum_{r} A_{k,t}(kn+t, r) x^r \frac{z^{kn+t}}{(kn+t)!} \qquad (t=0, 1, 2, \ldots)$$

are obtained in Theorems 1 and 2 below.

2. Let k, t be fixed integers, $k \geqq 2$ and $0 \leqq t < k$. Let $A_{k,t}(kn+t, r)$ denote the number of (k, t)-permutations (1.5) with r rises on the top line. A conventional rise on the left is counted. For example, with $k=3$, $t=1$, the (3, 1)-permutations



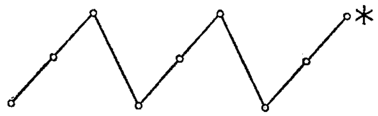


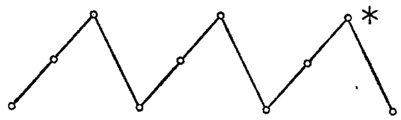
have 2 and 1 rises, respectively. It is convenient to take

$$A_{k,t}(j, r) = \delta_{r,0} \qquad (0 \leqq j \leqq t < k). \tag{2.1}$$

To begin with we set up a recurrence for $A_{k,t}(nk+t)$. Let π denote a typical (k, t)-permutation of Z_{kn+t} and consider the effect of removing the element $kn+t$. We take first the case $t=0$:

If the element kn is in the position marked with an asterisk, it is clear that π becomes a $(k, k-1)$-permutation of Z_{kn-1} and that the number of rises





has been decreased by 1. On the other hand, if kn is in any other position, π breaks into a $(k, k-1)$- and a $(k, 0)$-permutation. Moreover, because of the conventional rises, there is no loss in the number of rises. We accordingly get the recurrence

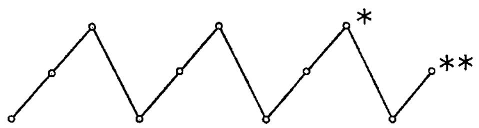
$$A_{k,0}(kn, r) = \sum_{j=1}^{n} \binom{kn-1}{kj-1} \sum_{s=0}^{r} A_{k,k-1}(kj-1, s)\, A_{k,0}(k(n-j), r-s) + A_{k,k-1}(kn-1, r-1) - A_{k,k-1}(kn-1, r). \tag{2.2}$$

For $t=1$, it is clear that π always breaks into a $(k, k-1)$- and a $(k, 1)$-permutation. There will be a loss of a rise only if $kn+1$ is in the position marked with an asterisk. Thus we get

$$A_{k,1}(kn+1, r) = \sum_{j=1}^{n} \binom{kn}{kj-1} \sum_{s=0}^{r} A_{k,k-1}(kj-1, s)\, A_{k,1}(k(n-j)+1, r-s) + kn\,[A_{k,k-1}(kn-1, r-1) - A_{k,k-1}(kn-1, r)]. \tag{2.3}$$

The coefficient kn is the number of ways of filling the extreme right hand position.

For $1<t<k$,



there are two exceptional positions for $kn+t$. In the first of the marked positions there is a loss of a rise. The resulting recurrence is

$$A_{k,t}(kn+t, r) = \sum_{j=1}^{n} \binom{kn+t-1}{kj-1} \sum_{s=0}^{r} A_{k,k-1}(kj-1, s)\, A_{k,t}(k(n-j)+t, r-s) + \binom{kn+t-1}{t} [A_{k,k-1}(kn-1, r-1) - A_{k,k-1}(kn-1, r)] + A_{k,t-1}(kn+t-1, r) \qquad (1<t<k). \tag{2.4}$$

3. Put

$$\tilde{A}_{k,t}(kn+t, x) = \sum A_{k,t}(kn+t, r)\, x^r. \tag{3.1}$$

Then (2.2), (2.3), (2.4) imply the following relations

$$\tilde{A}_{k,0}(kn, x) = \sum_{j=1}^{n} \binom{kn-1}{kj-1} \sum_{s=0}^{r} \tilde{A}_{k,k-1}(kj-1, x)\, \tilde{A}_{k,0}(k(n-j), x) - (1-x)\, \tilde{A}_{k,k-1}(kn-1, x), \tag{3.2}$$

$$\tilde{A}_{k,1}(kn+1, x) = \sum_{j=1}^{n} \binom{kn}{kj-1} \sum_{s=0}^{r} \tilde{A}_{k,k-1}(kj-1, x)\, \tilde{A}_{k,1}(k(n-j)+1, x) - kn(1-x)\, \tilde{A}_{k,k-1}(kn-1, x) \qquad (n \geqq 1), \tag{3.3}$$

$$\tilde{A}_{k,t}(kn+t, x) = \sum_{j=1}^{n} \binom{kn+t-1}{kj-1} \sum_{s=0}^{r} \tilde{A}_{k,k-1}(kj-1, x)\, \tilde{A}_{k,t}(k(n-j)+t-1, x) - \binom{kn+t-1}{t}(1-x)\tilde{A}_{k,k-1}(kn-1, x) + \tilde{A}_{k,t-1}(kn+t-1, x) \qquad 1 < t < k. \tag{3.4}$$

Next let

$$F_{k,t}(z) = \sum_{n=0}^{\infty} \tilde{A}_{k,t}(kn+t, x) \frac{z^{kn+t}}{(kn+t)!} \qquad (t \geqq 0). \tag{3.5}$$

For $t=0$, it follows from (3.2) that

$$F'_{k,0}(z) = \sum_{j=1}^{\infty} \tilde{A}_{k,k-1}(kj-1, x) \frac{z^{kj-1}}{(kj-1)!} \sum_{n=0}^{\infty} \tilde{A}_{k,0}(kn, x) \frac{z^{kn}}{(kn)!} - (1-x) \sum_{n=1}^{\infty} \tilde{A}_{k,k-1}(kn-1, x) \frac{z^{kn-1}}{(kn-1)!}.$$

Hence we get

$$F'_{k,0}(z) = F_{k,k-1}(z)\, F_{k,0}(z) - (1-x)\, F_{k,k-1}(z). \tag{3.6}$$

For $t=1$, it follows similarly from (3.3) that

$$F'_{k,1}(z) = F_{k,k-1}(z)\, F_{k,1}(z) - (1-x)\, zF_{k,k-1}(z) + 1. \tag{3.7}$$

The 1 on the extreme right corresponds to the term $A_{k,1}(1, x)$ on the left.

For $1<t<k$, if follows from (3.4) that

$$F'_{k,t}(z) = F_{k,k-1}(z)\, F_{k,t}(z) - (1-x) \frac{z^t}{t!} F_{k,k-1}(z) + F_{k,t-1}(z) \qquad (1<t<k). \tag{3.8}$$

4. We now consider the system of differential equations (3.6), (3.7), (3.8). It is convenient to transform the system by taking

$$F_{k,0}(z) = \frac{1}{\Phi_{k,0}(z)}, \tag{4.1}$$

$$F_{k,t}(z) = \frac{\Phi_{k,t}(z)}{\Phi_{k,0}(z)} \qquad (t \geqq 1). \tag{4.2}$$

Then (3.6), (3.7), (3.8) becomes

$$\Phi'_{k,0}(z)=-\Phi_{k,k-1}(z)+(1-x)\,\Phi_{k,k-1}(z)\,\Phi_{k,0}(z), \tag{4.3}$$

$$\Phi'_{k,1}(z)=\Phi_{k,0}(z)+(1-x)\,\Phi_{k,k-1}(z)\,\Phi_{k,1}(z)-(1-x)\,z\,\Phi_{k,k-1}(z), \tag{4.4}$$

$$\Phi'_{k,t}(z)=\Phi_{k,t-1}(z)+(1-x)\,\Phi_{k,k-1}\,\Phi_{k,t}(z) -(1-x)\frac{z^t}{t!}\,\Phi_{k,k-1}(z) \qquad (1<t<k), \tag{4.5}$$

respectively.

Put

$$\Phi(z)=1-(1-x)\,\Phi_{k,0}(z). \tag{4.6}$$

Then, by (4.3),

$$\Phi'(z)=(1-x)\,\Phi(z)\,\Phi_{k,k-1}(z), \tag{4.7}$$

so that, if y is an arbitrary function of z,

$$\left(\frac{y}{\Phi(z)}\right)'=\frac{1}{\Phi(z)}\left(y'-(1-x)\,\Phi_{k,k-1}(z)\,y\right).$$

Thus (4.3), (4.4), (4.5) become

$$\left(\frac{\Phi_{k,0}(z)}{\Phi(z)}\right)'=-\frac{\Phi_{k,k-1}(z)}{\Phi(z)}, \tag{4.8}$$

$$\left(\frac{\Phi_{k,1}(z)}{\Phi(z)}\right)'=\frac{\Phi_{k,0}(z)}{\Phi(z)}-(1-x)\,z\,\frac{\Phi_{k,k-1}(z)}{\Phi(z)}, \tag{4.9}$$

$$\left(\frac{\Phi_{k,t}(z)}{\Phi(z)}\right)'=\frac{\Phi_{k,t-1}(z)}{\Phi(z)}-(1-x)\frac{z^t}{t!}\,\frac{\Phi_{k,k-1}(z)}{\Phi(z)} \qquad (1<t<k). \tag{4.10}$$

If we put

$$\Psi_t(z)=\Psi_{k,t}(z)=\frac{\Phi_{k,t}(z)}{\Phi(z)} \qquad (t\geqq 0), \tag{4.11}$$

the equations (4.8), (4.9), (4.10) take on the simpler form

$$\Psi'_0(z)=-\Psi_{k-1}(z), \tag{4.12}$$

$$\Psi'_1(z)=\Psi_0(z)-(1-x)\,z\,\Psi_{k-1}(z), \tag{4.13}$$

$$\Psi'_t(z)=\Psi_{t-1}(z)-(1-x)\frac{z^t}{t!}\,\Psi_{k-1}(z) \qquad (1<t<k). \tag{4.14}$$

Now it is clear from (4.1), (4.2) and (4.11) that

$$\Psi_t(z)=\sum_{n=0}^{\infty} a_t(n)\,\frac{z^{kn+t}}{(kn+t)!}, \tag{4.15}$$

where $a_t(n) = a_t(n, x)$. Then, by (4.12), (4.13) and (4.14), we get

$$(4.16) \quad a_0(kn) \quad = -a_{k-1}(kn-1),$$

$$(4.17) \quad a_1(kn+1) = a_0(kn) - kn(1-x)\, a_{k-1}(kn-1),$$

$$(4.18) \quad a_t(kn+t) = a_{t-1}(kn+t-1) - \binom{kn+t-1}{t}(1-x)\, a_{k-1}(kn-1) \qquad (1<t<k).$$

Combining (4.16) and (4.17) we get

$$(4.19) \qquad a_1(kn+1) = -(1+kn(1-x))\, a_{k-1}(kn-1).$$

In (4.18) take $t=2$:

$$a_2(kn+2) = a_1(kn+1) - \binom{kn+1}{2}(1-x)\, a_{k-1}(kn-1)$$

$$= -\left[1+kn(1-x)+\binom{kn+1}{2}(1-x)\right] a_{k-1}(kn-1).$$

Next for $t=3$:

$$a_3(kn+3) = a_2(kn+2) - \binom{kn+2}{2}(1-x)\, a_{k-1}(kn-1)$$

$$= -\left[1+kn(1-x)+\binom{kn+1}{2}(1-x)+\binom{kn+2}{3}(1-x)\right] a_{k-1}(kn-1).$$

The general formula is evidently

$$(4.20) \qquad a_t(kn+t) = -[1+\sigma_t(n)(1-x)]\, a_{k-1}(kn-1) \qquad (1<t<k)$$

where

$$\sigma_t(n) = \sum_{j=1}^{t} \binom{kn+j-1}{j}.$$

In view of (4.19), this result holds for $t=1$ also.

In particular, for $t=k-1$, (4.20) becomes

$$a_{k-1}(kn+k-1) = -[1+\sigma_{k-1}(n)(1-x)]\, a_{k-1}(kn-1).$$

Hence

$$a_{k-1}(kn+k-1) = (-1)^n \prod_{j=0}^{n} [1+\sigma_{k-1}(j)(1-x)]\, a_{k-1}(k-1).$$

Since

$$\tilde{A}_{k-t,t}(k-1, x) = 1, \quad \Phi(0) = x, \quad a_{k-1}(k-1) = \frac{1}{x},$$

we have

$$(4.21) \qquad a_{k-1}(kn+k-1) = \frac{(-1)^n}{x} \prod_{j=0}^{n} [1+\sigma_{k-1}(j)(1-x)].$$

Hence, by (4.16) and (4.20)

$$a_0(kn)=\frac{(-1)^n}{x}\prod_{j=0}^{n-1}[1+\sigma_{k-1}(j)(1-x)], \tag{4.22}$$

$$a_t(kn+t)=\frac{(-1)^n}{x}[1+\sigma_t(n)(1-x)]\prod_{j=0}^{n-1}[1+\sigma_{k-1}(j)(1-x)] \qquad (1\leqq t<k). \tag{4.23}$$

Also, by (4.2) and (4.11),

$$F_{k,t}(z)=\frac{\Psi_t(z)}{\Psi_0(z)} \qquad (t\geqq 1), \tag{4.24}$$

while, by (4.1) and (4.6),

$$F_{k,0}(z)=1-x+\frac{\Psi_1}{\Psi_0(z)}. \tag{4.25}$$

To sum up, we state

Theorem 1. *Let $k\geqq 2$, $0\leqq t<k$. Then the generating functions $F_{k,t}(z)$ satisfy*

$$F_{k,0}(z)=1-x+\frac{x}{1+\sum_{n=1}^{\infty}(-1)^n\prod_{j=0}^{n-1}[1+\sigma_{k-1}(j)(1-x)]\frac{z^{kn}}{(kn)!}}, \tag{4.26}$$

$$F_{k,t}(z)=\frac{\frac{z^t}{t!}+\sum_{n=1}^{\infty}(-1)^n[1+\sigma_t(n)(1-x)]\prod_{j=1}^{n-1}[1+\sigma_{k-1}(j)(1-x)]\frac{z^{kn+t}}{(kn+t)!}}{1+\sum_{n=1}^{\infty}(-1)^n\prod_{j=0}^{n-1}[1+\sigma_{k-1}(j)(1-x)]\frac{z^{kn}}{(kn)!}} \tag{4.27}$$

$$(0<t<k).$$

5. The restriction $t<k$ in Theorem 1 can be removed. In defining $A_{k,t}(kn+t)$ for $t\geqq k$, we remark first that, when $t=k$, a possible rise may occur preceeding the extreme right hand element; however for $t>k$, such a rise is never counted. Clearly

$$A_{k,k}(kn+k)=A_{k,0}(kn+k). \tag{5.1}$$

We define $\tilde{A}_{k,t}(kn+t,\, x)$ and $F_{k,t}(z)$ by means of (3.1) and (3.5) for all $t\geqq 0$. Thus by (5.1) we have

$$F_{k,k}(z)=-1+F_{k,0}(z). \tag{5.2}$$

In the next place, for $t>k$, the recurrence (2.4) is valid. Then (3.4) is also valid and this in turn implies the truth of (3.8) and therefore of (4.10) and (4.14). Consequently (4.20) is also valid for $t>k$. It then follows that (4.27) holds for $t>k$. Moreover, it follows from (5.2) that (4.27) holds also for $t=k$.

We may accordingly state

Theorem 2. *Let* $k \geqq 2$, $t \geqq 0$. *Then the generating functions* $F_{k,t}(z)$ *satisfy* (4.26) *and* (4.27).

For $k=2$, it is easily verified that (4.26) and (4.27) are in agreement with the results of [**1**]. For $k=3$ we have

$$F_{3,0}(z)=1-x+\frac{x}{1+\sum_{n=1}^{\infty}(-1)^n\prod_{j=0}^{n-1}\left[1+9\binom{j+1}{2}(1-x)\right]\cdot\frac{z^{3n}}{(3n)!}}, \tag{5.3}$$

$$F_{3,1}(z)=\frac{z+\sum_{n=1}^{\infty}(-1)^n[1+3n(1-x)]\prod_{j=0}^{n-1}\left[1+9\binom{j+1}{2}(1-x)\right]\cdot\frac{z^{3n+1}}{(3n+1)!}}{1+\sum_{n=1}^{\infty}(-1)^n\prod_{j=0}^{n-1}\left[1+9\binom{j+1}{2}(1-x)\right]\cdot\frac{z^{3n}}{(3n)!}}, \tag{5.4}$$

$$F_{3,2}(z)=\frac{\frac{z^2}{2!}+\sum_{n=1}^{\infty}(-1)^n\prod_{j=1}^{n}\left[1+9\binom{j+1}{2}(1-x)\right]\cdot\frac{3^{3n+2}}{(3n+2)!}}{1+\sum_{n=1}^{\infty}(-1)^n\prod_{j=0}^{n-1}\left[1+9\binom{j+1}{2}(1-x)\right]\cdot\frac{z^{3n}}{(3n)!}}. \tag{5.5}$$

Thus

$$F_{3,0}(z)=1+x\frac{z^3}{3!}+(10x+9x^2)\frac{z^6}{6!}+\dots . \tag{5.6}$$

The ten permutations with $r=1$ are:

126345, 136245, 146235, 156234, 236145, 246135, 256134, 346125, 356124, 456123.

The nine permutations with $r=2$ are:

124356, 125346, 134256, 135246, 145236, 234156, 235146, 245136, 345136.

$$F_{3,1}(z)=z+3x\frac{z^4}{4!}+(42x+54x^2)\frac{z^7}{7!}+\dots , \tag{5.7}$$

$$F_{3,2}(z)=\frac{z^2}{2!}+9x\frac{z^5}{5!}+(234x+243x^2)\frac{z^8}{8!}+\dots . \tag{5.8}$$

We shall not take the space to list the permutations corresponding to the terms in z^7 and z^8 in (5.7) and (5.8). The coefficient $9x$ in (5.8) corresponds to the permutations

12435, 12534, 13425, 13524, 14523, 23415, 23514, 24513, 34512.

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