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450. VARIATIONS AND GENERALIZATIONS OF AN INEQUALITY DUE TO BOHR*

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Editorial Committee

P. M. Vasić and J. D. Kečkić [1] have generalized an inequality of Bohr (see, for example, [2] p. 61 or [3] p. 312)

$$|z_1+z_2|^2 \leqq (1+c)\,|z_1|^2+\left(1+\frac{1}{c}\right)|z_2|^2,$$

where z_1 and z_2 are complex numbers and $c>0$. Their statement reads:

Let $z_1,\dots,z_n$ be complex numbers, and $p_1,\dots,p_n$ positive numbers. Then, for $r>1$,

$$\left|\sum_{i=1}^{n} z_i\right|^r \leqq \left(\sum_{i=1}^{n} p_i^{\frac{1}{1-r}}\right)^{r-1} \sum_{i=1}^{n} p_i\,|z_i|^r. \tag{1}$$

Since inequality (1) is based on the properties of the function $x\mapsto x^r\,(r>1)$, it is natural to investigate whether an analogous inequality is valid, if, instead of that function, a more general function f, which has some, but not all, properties of function $x\mapsto x^r$ is considered.

In the present paper we shall prove inequality (9) which generalizes (1). We shall also prove the inequality, complementary to (9).

In further text, $z=(z_1,\dots,z_n)$ will denote a complex sequence, $p=(p_1,\dots,p_n)$ a positive sequence, and I the interval $[0,+\infty)$.

Theorem 1. *Let f be a strictly convex function on I and let*

$$f(uv)\leqq f(u)f(v)\quad (u,v\in I);\quad \lim_{t\to 0+}\frac{f(t)}{t}=0;\quad \lim_{t\to+\infty}\frac{f(t)}{t}=+\infty. \tag{2}$$

Then

$$f\left(\sum_{i=1}^{n} x_i\right)\leqq g\left(\sum_{i=1}^{n}\frac{1}{g^{-1}(p_i)}\right)\sum_{i=1}^{n} p_i f(x_i), \tag{3}$$

where $x_i\in I$ $(i=1,\dots,n)$ and $g(t)=\dfrac{f(t)}{t}$.

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Proof. From the hypotheses of the theorem it follows directly that $f(0)=0$ and that function g, defined by $g(t)=\frac{f(t)}{t}$, is increasing for $t>0$. It means that there exists the function g^{-1}, inverse to g. Therefore, since $\lim_{t\to 0+} g(t)=0$ and $\lim_{t\to+\infty} g(t)=+\infty$, we conclude that equality $g(x)=y$ has a unique solution with respect to x for every $y>0$.

For any convex function f, the inequality

$$f\left(\frac{\sum_{i=1}^{n} q_i y_i}{\sum_{i=1}^{n} q_i}\right)\leq\frac{\sum_{i=1}^{n} q_i f(y_i)}{\sum_{i=1}^{n} q_i} \tag{4}$$

is valid, for arbitrary numbers $q_i>0$, $y_i\geq 0$ $(i=1,\ldots,n)$. On the other hand, having in view the inequality in (2) we have

$$f\left(\sum_{i=1}^{n} q_i y_i\right)=f\left(\frac{\sum_{i=1}^{n} q_i y_i}{\sum_{i=1}^{n} q_i}\sum_{i=1}^{n} q_i\right)\leq f\left(\frac{\sum_{i=1}^{n} q_i y_i}{\sum_{i=1}^{n} q_i}\right)f\left(\sum_{i=1}^{n} q_i\right). \tag{5}$$

From (4) and (5) it follows

$$f\left(\sum_{i=1}^{n} q_i y_i\right)\leq g\left(\sum_{i=1}^{n} q_i\right)\sum_{i=1}^{n} q_i f(y_i). \tag{6}$$

If we introduce the substitution $q_i y_i=x_i$ $(i=1,\ldots,n)$ and again apply the inequality in (2), we find

$$f\left(\sum_{i=1}^{n} x_i\right)\leq g\left(\sum_{i=1}^{n} q_i\right)\sum_{i=1}^{n} q_i f\left(\frac{x_i}{q_i}\right)\leq g\left(\sum_{i=1}^{n} q_i\right)\sum_{i=1}^{n} g\left(\frac{1}{q_i}\right)f(x_i). \tag{7}$$

Let $g\left(\frac{1}{q_i}\right)=p_i$ $(i=1,\ldots,n)$. Then $q_i=\frac{1}{g^{-1}(p_i)}$ $(i=1,\ldots,n)$ and (7) becomes

$$f\left(\sum_{i=1}^{n} x_i\right)\leq g\left(\sum_{i=1}^{n}\frac{1}{g^{-1}(p_i)}\right)\sum_{i=1}^{n} p_i f(x_i),$$

which is, in fact, inequality (3).

Remark 1. If conditions (2) are replaced by

$$f(uv)\geq f(u)f(v)\quad (u,v\in I);\quad \lim_{t\to 0+}\frac{f(t)}{t}=+\infty;\quad \lim_{t\to+\infty}\frac{f(t)}{t}=0,$$

inequality

$$f\left(\sum_{i=1}^{n} x_i\right)\geq g\left(\sum_{i=1}^{n}\frac{1}{g^{-1}(p_i)}\right)\sum_{i=1}^{n} p_i f(x_i) \tag{8}$$

holds for f concave.

Theorem 2. *Let function f satisfy the conditions of theorem* 1. *Then, for the sequences z and p we have*

$$f\left(\left|\sum_{i=1}^{n} z_i\right|\right) \leqq g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{n} p_i f(|z_i|). \tag{9}$$

Proof. From the hypotheses of the theorem it follows directly that f is an increasing function. Using the triangle inequality $\left|\sum_{i=1}^{n} z_i\right| \leqq \sum_{i=1}^{n} |z_i|$, from theorem 1 it follows

$$f\left(\left|\sum_{i=1}^{n} z_i\right|\right) \leqq f\left(\sum_{i=1}^{n} |z_i|\right) \leqq g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{n} p_i f(|z_i|),$$

which is, in fact, inequali y (9).

REMARK 2. From the proof of theorem 2 it is clear that inequality (9) ho!ds for the e!ements of an arbitrary normed vector space. Namely, if V is a normed vector space with the norm $\|\cdot\|$ and if function f satisfies conditions of theorem 1, then inequality

$$f\left(\left\|\sum_{i=1}^{n} x_i\right\|\right) \leqq g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{n} p_i f(\|x_i\|)$$

is valid, where $x_1, \ldots, x_n \in V$.

Using the complementary triangle inequality, we may derive a theorem which corres;onds to the result of remark 1, in the same sen e in which theorem 2 corresponds to theorem 1. Namely, we have

Theorem 3. *Let the function f satisfy conditions of remark* 1 *and let f be a nondecreasing function. Then for the sequences z and p, we have*

$$f\left(C\left|\sum_{i=1}^{n} z_i\right|\right) \geqq g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{n} p_i f(|z_i|), \tag{10}$$

where

$$C = \begin{cases} \dfrac{1}{\cos\theta} & \left(\alpha - \theta \leqq \arg z_i \leqq \alpha + \theta;\quad 0 < \theta < \dfrac{\pi}{2}\right), \\ \dfrac{1}{\max\left(\dfrac{\sqrt{2}}{2},\ \cos\theta\right)} & \left(\alpha \leqq \arg z_i \leqq \alpha + \theta;\quad 0 < \theta < \dfrac{\pi}{2}\right), \\ +\infty & (\textit{in other cases}), \end{cases} \tag{11}$$

where, owing to the monotony of f, we agree to take $f(+\infty) = +\infty$.

Proof. If the substitution $x_i = |z_i|$ $(i = 1, \ldots, n)$ is in roduced in o (8), we get

$$f\left(\sum_{i=1}^{n} |z_i|\right) \geqq g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{n} p_i f(|z_i|). \tag{12}$$

Using the complementary triangle inequality, if C is given by (11) (see [4], [5], [6]), we find

$$C\left|\sum_{i=1}^{n} z_i\right| \geqq \sum_{i=1}^{n} |z_i|,$$

wherefrom, since f is a nondecreasing function, inequality (12) follows.

REMARK 3. Inequalities similar to (10) may be obtained for the vectors in HILBERT's and BANACH's space (see [7] and [5]).

In the case when $f(t)=t^r\,(r>1)$ inequality (9) is reduced to (1.1) from [1].

If $g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right)=1$ inequality (9) becomes

$$f\left(\left|\sum_{i=1}^{n} z_i\right|\right) \leqq \sum_{i=1}^{n} p_i f(|z_i|),$$

which for $f(t)=t^r\,(r>1)$ yields inequality (1.7) from [1].

If substitution $p_i=g(n)\,(i=1,\dots,n)$ is introduced into (9), we get

$$f\left(\left|\sum_{i=1}^{n} z_i\right|\right) \leqq f(1)\frac{f(n)}{n} \sum_{i=1}^{n} f(|z_i|).$$

The above inequality holds for all functions f satisfying conditions of theorem 1, and reduces to (1.8) from [1], when $f(t)=t^r\,(r>1)$.

REMARK 4. Similarly to the proof of theorem 1, the inequality

$$f\left(\int_a^b q(x)\,dx\right) \leqq g\left(\int_a^b \frac{1}{g^{-1}(p(x))}\,dx\right) \int_a^b p(x) f(q(x))\,dx$$

may be proved, where $p(x)>0$, $x\in[a, b]$, and where f satisfies conditions of theorem 1. (In this case q is an arbitrary nonnegative function on $[a, b]$, $a\geqq 0$).

We shall now prove some theorems regarding the lower bound of the expression $f\left(\sum_{i=1}^{n} x_i\right)$ in (3), i.e. the lower bound of $f\left(\left|\sum_{i=1}^{n} z_i\right|\right)$ in (9). Thus, in the following theorem we give complementary inequalities for the above mentioned inequality.

Theorem 4. *Let the function f satisfy conditions of theorem 1. Then, for the real sequence p, for which* $g^{-1}\left(\frac{1}{p_i}\right)\geqq 1$,

$$\sum_{i=1}^{n} p_i f(x_i) \leqq f\left(\sum_{i=1}^{n} x_i\right) \qquad (x_i\in I\,(i=1,\dots,n)). \tag{13}$$

Proof. Since $f(0)=0$, in virtue of an inequality due to M. PETROVIĆ (see (2) in [8]), we have $\sum_{i=1}^{n} q_i f(y_i) \leqq f\left(\sum_{i=1}^{n} q_i y_i\right)$, where $q_i\geqq 1$, $y_i\geqq 0\,(i=1,\dots,n)$.

Furthermore, after substitution $x_i = q_i y_i$ we get

$$f\left(\sum_{i=1}^{n} x_i\right) \geq \sum_{i=1}^{n} q_i f\left(\frac{x_i}{q_i}\right) = \sum_{i=1}^{n} q_i g(q_i) f\left(\frac{x_i}{q_i}\right) \frac{1}{g(q_i)}$$
$$= \sum_{i=1}^{n} f(q_i) f\left(\frac{x_i}{q_i}\right) \frac{1}{g(q_i)} \geq \sum_{i=1}^{n} f(x_i) \frac{1}{g(q_i)}.$$

If substitution $\frac{1}{g(q_i)} = p_i$ is introduced into preceding inequality, (13) is obtained, and thereby the theorem is proved.

A direct consequence of theorem 4 is

Theorem 5. *Let the function f satisfy conditions of theorem* 1. *Let p be real sequence such that $g^{-1}\left(\frac{1}{p_i}\right) \geq 1$. Then*

$$\frac{1}{f(C)} \sum_{i=1}^{n} p_i f(|z_i|) \leq f\left(\left|\sum_{i=1}^{n} z_i\right|\right) \leq g\left(\sum_{i=1}^{n} \frac{1}{g^{-1}(p_i)}\right) \sum_{i=1}^{n} p_i f(|z_i|), \tag{14}$$

where C is defined by (11).

We shall prove only the left-hand side of (14).

If substitution $x_i = |z_i|$ $(i = 1, \ldots, n)$ is introduced into (13), we get

$$\sum_{i=1}^{n} p_i f(|z_i|) \leq f\left(\sum_{i=1}^{n} |z_i|\right) \leq f\left(C\left|\sum_{i=1}^{n} z_i\right|\right) \leq f(C) f\left(\left|\sum_{i=1}^{n} z_i\right|\right),$$

wherefrom (14) follows.

Remark 5. Inequalities similar to (13) and (14) may be obtained for concave functions as well as for vectors of Hilbert's, or Banach's space.

Examples. Conditions of theorem 1 are fulfilled, for example, by the function $t \mapsto f(t) = t^r \exp \frac{t}{t+s}$ $(r > 1,\ s > 0)$. In this case inequality (9) becomes

$$\left|\sum_{i=1}^{n} z_i\right|^r \exp \frac{\left|\sum_{i=1}^{n} z_i\right|}{s + \left|\sum_{i=1}^{n} z_i\right|} \leq A \sum_{i=1}^{n} c_i^{1-r} |z_i|^r \exp\left(\frac{1}{sc_i + 1} + \frac{|z_i|}{s + |z_i|}\right), \tag{15}$$

where $A = \left(\sum_{i=1}^{n} c_i\right)^{r-1} \exp \frac{\sum_{i=1}^{n} c_i}{s + \sum_{i=1}^{n} c_i}$ and c_i $(i = 1, \ldots, n)$ are positive numbers.

In the case when $s = 1$, $c_1 = \cdots = c_n = 1$ inequality (15) becomes

$$\left|\sum_{i=1}^{n} z_i\right|^r \exp \frac{\left|\sum_{i=1}^{n} z_i\right|}{1 + \left|\sum_{i=1}^{n} z_i\right|} \leq n^{r-1} \exp\left(\frac{n}{n+1} + \frac{1}{2}\right) \sum_{i=1}^{n} |z_i|^r \exp \frac{|z_i|}{1 + |z_i|}.$$

Again, if we take $s=n$, $c_1=\cdots=c_n=1$ inequality (15) becomes

$$\left|\sum_{i=1}^{n} z_i\right|^r \exp\frac{\left|\sum_{i=1}^{n} z_i\right|}{n+\left|\sum_{i=1}^{n} z_i\right|}\le n^{r-1}\exp\left(\frac{1}{n+1}+\frac{1}{2}\right)\sum_{i=1}^{n}|z_i|^r\exp\frac{|z_i|}{n+|z_i|}.$$

Setting $s=1$, $c_1=\cdots=c_n=\frac{1}{n}$ into (15), we get

$$\left|\sum_{i=1}^{n} z_i\right|^r \exp\frac{\left|\sum_{i=1}^{n} z_i\right|}{1+\left|\sum_{i=1}^{n} z_i\right|}\le n^{r-1}\exp\left(\frac{n}{n+1}+\frac{1}{2}\right)\sum_{i=1}^{n}|z_i|^r\exp\frac{|z_i|}{1+|z_i|}.$$

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