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449. AN ITERATIVE SOLUTION OF ALGEBRAIC EQUATIONS WITH A PARAMETER TO ACCELERATE CONVERGENCE*

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An algorithm for solving an algebraic equation by an iterative method with a parameter to accelerate convergence is given in this paper. Introduction of this parameter slightly increases the number of operations in an iterative step and considerably decreases the number of iterative steps in root calculation.

1. Introduction

For computer-aided calculation of the roots of numerical algebraic equations the most suitable are iterative methods. They converge slowly if the initial approximate root value is too far from its accurate value. This paper proposes an iterative method with a parameter to accelerate convergence. Dynamic modification of the parameter value decreases the number of the steps needed to get the approximate root value with desired accuracy.

2. Solution of algebraic equations

Let there be given the equation

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n, \tag{1}$$

where the coefficients a_i $(i = 0, 1, \ldots, n)$ are real. Consider the iterative procedure

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i) + f(x_i)\dfrac{\varphi'(x_i)}{\varphi(x_i)}}, \tag{2}$$

where $\varphi(x)$ is an arbitrary continuous and twice differentiable function, whose zeros do not coincide with roots of equation (1) [1].

If one substitutes $f(x_i)/f'(x_i) = \Delta(x_i)$, $\varphi'(x_i)/\varphi(x_i) = Q(x_i)$, formula (2) becomes

$$x_{i+1} = x_i - \frac{\Delta(x_i)}{1 + \Delta(x_i) Q(x_i)}. \tag{3}$$

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Formula (2), i.e. (3), is generalized NEWTON's formula. After evident identical transformations formula (3) becomes

$$(4)\qquad x_{i+1}=x_i-\Delta(x_i)+\frac{\Delta(x_i)^2 Q(x_i)}{1+\Delta(x_i)Q(x_i)}.$$

Analysing formula (4), one may conclude that generalized NEWTON's formula differs from the NEWTON-RAPHSON's formula in the term $\Delta(x_i)^2 Q(x_i)/(1+$ $+\Delta(x_i)Q(x_i))$ the value of which, when $|\Delta(x_i)|$ is large, increases correction to form the next approximate root value. When $\Delta(x_i)\to 0$ formula (4) becomes the NEWTON-RAPHSON's formula. If one makes the substitution of $\varphi(x)\equiv x^p$ in (2), where p is a real, (for the present) unknown parameter or argument function x, formula (2) acquires the form

$$(5)\qquad x_{i+1}=x_i\left(1-\frac{f(x_i)}{x_i f'(x_i)+pf(x_i)}\right).$$

Parameter p may be chosen so that for the fixed value x_1, denominator $x_i f'(x_i)+p\,f(x_i)$ in formula (5) has as small value as possible.

For $p=-n$ the coefficient of x_i^n is annullated in the denominator $x_i f'(x_i)+$ $+pf(x_i)$, and for $p=-n+\frac{a_1}{a_0 x_i}$ the coefficients of x_i^n and x_i^{n-1} are annullated. In practice, in the first iteration for p, the value $p_0=1-n$ is taken. Further values of p_i are formed by the relation $p_i=p_{i-1}-1.5$, until the condition $i\leqq n-2$ is valid, and after that the values $p_i=\frac{1}{2}p_{i-1}$ are accepted.

For $p=1-n$ formula (5) becomes

$$x_{i+1}=x_i\left(1-\frac{f(x_i)}{x_i f'(x_i)+(1-n)f(x_i)}\right),$$

which is in [2] obtained by means of a very complex mathematical calculation.

Formula (5) for $p=0$ is reduced to NEWTON-RAPHSON's formula. In NEWTON-RAPHSON's formula correction of $\Delta(x_i)$ is additive. In generalized NEWTON's method the correction of $\delta(x_i)=1-f(x_i)/(x_i f'(x_i)+pf(x_i))$ is multiplicative. The more you estimate the larger root per modulus the more it becomes larger. The rapid convergence through formula (5) is guaranteed if the roots of equation (1) are calculated starting from per modulus the largest to per modulus the smallest one. If the largest positive root is calculated, the initial approximate root value should be larger than that of the root, and if the smallest negative root is estimated the initial approximate value should be smaller than that of the root.

All these considerations hold if the roots per modulus are larger than unity. For the roots per modulus smaller than unity, the substitution of $y=1/x$ is made in (1). Through it one can provide rapid convergence for such roots as well.

3. Comparison of methods

Iterative formula (5) is slightly more complex than NEWTON-RAPHSON's formula. To calculate the next approximate root value of the equation (1) in one iterative step in NEWTON-RAPHSON's formula one must do $2n$ additions (subtractions) and $2n-1$ multiplications and in formula (5) $2n+1$ additions and $2n+1$ multiplications. However, it has got a number of practical advantages. Square convergence is realized already in the first iterations, even when the initial approximate root value is far from being accurate. If in one iteration the number of accurate root digit is k, in NEWTON-RAPHSON's formula in the next iteration it will be $2k$, and in formula (5) $2k+1$. A large number of examples have shown that $l=1$ in most cases and very rarely it is $l=2$.

Let us designate with ts the computer time necessary for an addition (subtraction), and with im computer time for a multiplication. Let lt_2 and lg_2 be the number of iterations according to NEWTON-RAPHSON's and generalized method for calculation of a single root with accuracy of 10^{-t} decimals, respectively. The use of formula (5) is more suitable than the use of NEWTON-RAPHSON's formula, if following inequality holds:

$$lt_2 > \frac{2(n+1)lg_2}{2n-1}. \tag{6}$$

Inequality (6) will be satisfied already in equations of the fifth degree if the number of iterations, according to formula (5), is less by a fourth than the number of iterations according to NEWTON-RAPHSON's formula. If the savings in the number of iterations according to formula (5) is 50% in relation to the number of iterations in NEWTON-RAPHSON's formula then the inequality (6) holds in the quadratic equation too.

4. Program

The FORTRAN language program has been made for the described method. The same has been tested on the IBM 1130 computer at the Faculty of Electronic Engineering at Niš for calculation of the real roots of equations up to sixteenth degree. For the initial root values taken were $x_0=a_1/a_0$ if $a_1\neq 0$, and $x_0=\sqrt{-2a_2/a_0}$ if $a_1=0$. Each root has been found within the accuracy of the computer for less than 20 iterations.

5. Numerical example

Iteration steps for the calculation of the largest root of equation

$$x^6-21x^5+175x^4-735x^3+1624x^2-1764x+720=0 \tag{7}$$

with accuracy 10^{-9} using NEWTON-RAPHSON's formula and formula (5) are tabulated in Table 1. Both formulas start with the same initial value $x_0=21$. The accurate root value is 6.

No. of iterations	NEWTON-RAPHSON's formula	Formula (5)
1	18.1113070912	11.7477735025
2	15.7096973620	6.9376600979
3	13.7151463132	6.2206861840
4	12.0612197765	6.0211908274
5	10.6928759581	6.0001895653
6	9.5646603031	6.0000000085
7	8.6392440360	6.0000000016
8	7.8862786929	
9	7.2815684100	
10	6.8066024619	
11	6.4485083985	
12	6.2001661620	
13	6.0578467745	
14	6.0065370576	
15	6.0000957447	
16	6.0000000189	
17	5.9999999990	
18	5.9999999998	

Table 1

Table 2 shows the calculated root values of equation (7) with accuracy 10^{-9} and the number of iterations according to NEWTON-RAPHSON's formula and formula (5). The accurate root values are $x_i = i\,(i = 1, 2, 3, 4, 5, 6)$.

NEWTON-RAPHSON		Formula (5)	
Root value	No. of iterations	Root value	No of iterations
5.99999999	18	6.00000000	7
5.00000000	14	4.99999995	7
3.99999996	12	4.00000007	6
3.00000005	9	2.99999993	5
1.99999996	6	2.00000003	5
1.00000000	1	0.99999999	1

Table 2

6. Conclusion

Using the described algorithm one can considerably decrease the number of iterative steps with a somewhat greater number of operations in one iterative step. It is suitable for calculation of the roots of algebraic equations when the initial approximate root values are unknown. Since the roots are calculated successively, the last calculated root is obtained with the greatest error. If greater accuracy is desired, the obtained solutions are taken as the initial values and so better approximate values are obtained directly from equation (1).

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