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443. INEQUALITIES FOR THE TRIANGLE*

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Theorem 1. *For the triangle the following inequality*

(1) $$\frac{m_a}{b+c}+\frac{m_b}{c+a}+\frac{m_c}{a+b}\geqq\frac{3\sqrt{3}}{2}\,\frac{r}{R}$$

holds, with equality if and only if the triangle is equilateral.

Proof. Let a triangle ABC be given, and let $m_c=CC_1$. From triangles ACC_1 and BCC_1 we have respectively

$$\frac{R\sin\gamma}{\sin\sphericalangle ACC_1}=\frac{m_c}{\sin\alpha},\qquad \frac{R\sin\gamma}{\sin\sphericalangle BCC_1}=\frac{m_c}{\sin\beta}.$$

Therefrom

$$\sin\gamma\,(\sin\alpha+\sin\beta)\frac{R}{m_c}=\sin\sphericalangle ACC_1+\sin\sphericalangle BCC_1\leqq 2\sin\frac{\gamma}{2}$$

or

(2) $$\frac{2m_c}{a+b}\geqq\cos\frac{\gamma}{2}.$$

Analogously to inequality (2) we have

(3) $$\frac{2m_a}{b+c}\geqq\cos\frac{\alpha}{2},\qquad \frac{2m_b}{c+a}\geqq\cos\frac{\beta}{2}.$$

On the basis of (2) and (3)

(4) $$2\sum\frac{m_a}{b+c}\geqq\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}.$$

Since ([**1**])

(5) $$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\geqq\sin\alpha+\sin\beta+\sin\gamma$$

and

$$a+b+c\geqq 6r\sqrt{3},$$

i. e.,

(6) $$\sin\alpha+\sin\beta+\sin\gamma\geqq 3\frac{r}{R}\sqrt{3},$$

we get the required inequality.

* Presented June 23, 1973 by O. Bottema and R. R. Janić.

Theorem 2. *The following inequality*

$$\frac{\sqrt{3}}{F} \geqq \frac{1}{m_b m_c} + \frac{1}{m_c m_a} + \frac{1}{m_a m_b}$$

holds, with equality only for an equilateral triangle.

Proof. From (2) we have

$$2m_c \geqq (a+b)\cos\frac{\gamma}{2} \Rightarrow m_c^2 \geqq ab\cos^2\frac{\gamma}{2}$$

i.e.,

(7) $$\sqrt{\operatorname{tg}\frac{\gamma}{2}} \geqq \frac{\sqrt{F}}{m_c}.$$

Analogously we get

(8) $$\sqrt{\operatorname{tg}\frac{\alpha}{2}} \geqq \frac{\sqrt{F}}{m_a}, \qquad \sqrt{\operatorname{tg}\frac{\beta}{2}} \geqq \frac{\sqrt{F}}{m_b}.$$

From (7) and (8) we have

(9) $$\sum\sqrt{\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}} \geqq F\sum\frac{1}{m_b m_c}.$$

Since

(10) $$\left(\sum\sqrt{\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}}\right)^2 \leqq 3\sum\operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2} = 3,$$

from (9) we obtain the requested inequality.

Theorem 3.

$$(m_b m_c)^2 + (m_c m_a)^2 + (m_a m_b)^2 \geqq rs^2(4R+r).$$

Equality holds if and only if the triangle is equilateral.

Proof. Since

$$m_a^2 = \frac{b^2+c^2}{2} - \frac{a^2}{4} = \frac{2(b^2+c^2)-a^2}{4},$$

and

$$b^2 + c^2 \geqq 2bc,$$

we obtain

$$m_a^2 \geqq s(s-a).$$

Similarly

$$m_b^2 \geqq s(s-b), \qquad m_c^2 \geqq s(s-c).$$

Therefore

(11) $$m_b^2 m_c^2 \geqq s^2(s-b)(s-c) = F^2\frac{s}{s-a} = Fsr_a.$$

Similarly, we get

(12) $$m_c^2 m_a^2 \geqq Fsr_b, \qquad m_a^2 m_b^2 \geqq Fsr_c.$$

Adding (11) and (12)

$$\sum(m_b m_c)^2 \geqq Fs(r_a + r_b + r_c) = rs^2(4R+r).$$

Theorem 4.

$$\sum \frac{a^2}{m_b^2 + m_c^2} \leqq 2,$$

with equality if and only if the triangle is equilateral.

Proof. This inequality is equivalent to

$$\sum \left(1 + \frac{1}{4} \frac{b^2 + c^2}{a^2}\right)^{-1} \leqq 2. \tag{13}$$

Start with the function

$$f(x) = \frac{x}{mx + n} \quad (m,\ n,\ x > 0). \tag{14}$$

Since

$$f''(x) = \frac{-2mn}{(mx + n)^3} < 0,$$

function f, given by (14) is concave, so that

$$\sum_{i=1}^{3} \frac{1}{m + \frac{n}{x_i}} \leqq 3\left(m + 3 \frac{n}{x_1 + x_2 + x_3}\right)^{-1}. \tag{15}$$

Putting in (15) $m = 3$, $n = a^2 + b^2 + c^2$, $x_1 = a^2$, $x_2 = b^2$, $x_3 = c^2$ we get inequality (13).

Theorem 5.

$$\sum \frac{b^2 + c^2}{w_a^2} \geqq 8.$$

Equality holds only for an equilateral triangle.

Proof. Since

$$\cos \frac{\gamma}{2} = \frac{w_c}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \geqq \frac{2 w_c}{a + b},$$

we have

$$\frac{a + b}{w_c} \geqq 2 \sec \frac{\gamma}{2} \Rightarrow \frac{(a + b)^2}{w_c^2} \geqq 4 \sec^2 \frac{\gamma}{2}. \tag{16}$$

From $2(a^2 + b^2) \geqq (a + b)^2$ it follows

$$\frac{a^2 + b^2}{w_c^2} \geqq 2 \sec^2 \frac{\gamma}{2}. \tag{17}$$

Similarly

$$\frac{b^2 + c^2}{w_a^2} \geqq 2 \sec^2 \frac{\alpha}{2}, \quad \frac{c^2 + a^2}{w_b^2} \geqq 2 \sec^2 \frac{\beta}{2}. \tag{18}$$

Adding up inequalities (17) and (18) we get

$$\sum \frac{a^2 + b^2}{w_c^2} \geqq 2 \sum \sec^2 \frac{\alpha}{2} \geqq 8.$$

REFERENCE

1. M. S. Klamkin: *Notes on inequalities involving triangles or tetrahedrons.* These Publications **№ 330—№ 337** (1970), 1—15.