PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 412 — № 460 (1973)

420. SOME NEW RELATIONS FOR JACOBI POLYNOMIALS ARISING FROM COMMUNICATION NETWORKS THEORY*

Branko D. Raković and *Petar M. Vasić*

0. The classical orthogonal polynomials are widely used in communication theory and particularly in the synthesis of electric filters. Apart from CHEBYSHEV polynomials, which are of utmost importance in the synthesis of filters exhibiting a sharp increase in attenuation as the frequency increases above cutoff, other classes of orthogonal polynomials such as JACOBI, LEGENDRE, GEGENBAUER, LAGUERRE, HERMITE and BESSEL polynomials have found many useful applications in the synthesis of electric filters.

A special class of filter functions of odd order providing monotonic magnitude characteristic of the resulting filter has first been investigated by PAPOULIS [1] by means of LEGENDRE polynomials. Subsequently these results have been extended so as to include filters of even degree [2], [3], and also some other functions leading to the same class of filtering networks whose magnitude response is bounded to be monotonic have been derived using a different approach based on the applications of JACOBI polynomials [4].

In the first part of this paper a new summation formula for Jacobi polynomials is derived, the special cases of which enable the solution for the transfer functions of low-pass filters with monotonic magnitude response to be obtained in more compact form for both n even and n odd simultaneously. In the second part a general differential-difference property of the JACOBI polynomials is derived the special case of which stems from some properties of the aforementioned class of monotonic magnitude filters.

1. We shall start with the known relationship for the JACOBI polynomials [5; p. 276]

$$(2n-2k+\lambda-1)P_{n-k}^{(\alpha-1,\beta)}(x)$$
$$=(n-k+\lambda-1)\,P_{n-k}^{(\alpha,\beta)}(x)-(n-k+\beta)\,P_{n-k-1}^{(\alpha,\beta)}(x) \qquad (\lambda=\alpha+\beta+1)$$

which is valid for $\alpha>0$, $\beta>-1$, $k\leqq n-1$.

* Received May 20, 1973.

Multiplying both sides by $(\lambda)_{n-k-1}$ $(n-k+\beta+1)_k$ $(\neq 0)$ where

$$(a)_k = a(a+1)\cdots(a+k-1) \qquad (k \geqq 1), \quad (a)_0 = 1,$$

and summing with respect to k $(k=0, 1, \ldots, n-1)$, we find

$$\sum_{k=0}^{n-1} (2n-2k+\lambda-1)(\lambda)_{n-k-1}(n-k+\beta+1)_k P_{n-k}^{(\alpha-1,\beta)}(x)$$

$$= \sum_{k=0}^{n-1} (\lambda)_{n-k-1}(n-k+\lambda-1)(n-k+\beta+1)_k P_{n-k}^{(\alpha,\beta)}(x)$$

$$- \sum_{k=0}^{n-1} (\lambda)_{n-k-1}(n-k+\beta)(n-k+\beta+1)_k P_{n-k-1}^{(\alpha,\beta)}(x)$$

$$= \sum_{r=0}^{n-1} (\lambda)_{n-r}(n-r+\beta+1)_r P_{n-r}^{(\alpha,\beta)}(x)$$

$$- \sum_{r=1}^{n} (\lambda)_{n-r}(n-r+\beta+1)_r P_{n-r}^{(\alpha,\beta)}(x)$$

wherefrom follows

$$\sum_{k=0}^{n-1} (2n-2k+\lambda-1)\ (\lambda)_{n-k-1}(n-k+\beta+1)_k P_{n-k}^{(\alpha-1,\beta)}(x)$$

$$= (\lambda)_n P_n^{(\alpha,\beta)}(x) - (\beta+1)_n P_0^{(\alpha,\beta)}(x).$$

Now with the change of summation limits and with the summation performed with respect to k $(k=1, 2, \ldots, n)$ we finally get

$$\sum_{k=1}^{n} (2k+\lambda-1)\ (\lambda)_{k-1}(k+\beta+1)_{n-k} P_k^{(\alpha-1,\beta)}(x) = (\lambda)_n P_n^{(\alpha,\beta)}(x) - (\beta+1)_n, \tag{1.1}$$

where, as before, $\alpha > 0$, $\beta > -1$, $\lambda = \alpha+\beta+1$.

Special cases. For $\alpha=1$, $\beta=q-1$ $(q>0)$, we have $\lambda=q+1$ and the summation formula (1.1) reduces to

$$\sum_{k=1}^{n} (2k+q) P_k^{(0,q-1)}(x) = (n+q) P_n^{(1,q-1)}(x) - q,$$

so that

$$\sum_{k=0}^{n} (2k+q) P_k^{(0,q-1)}(x) = (n+q) P_n^{(1,q-1)}(x) \qquad (q>0). \tag{1.2}$$

Using the relationship between Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and the shifted Jacobi polynomials $G_n(p, q, x)$

$$P_n^{(p-q,q-1)}(x) = \frac{\Gamma(2n+p)}{n!\,\Gamma(n+p)} G_n\left(p, q, \frac{1+x}{2}\right)$$

we easily deduce from (1.2)

$$(1.3)\qquad \sum_{k=0}^{n} \frac{\Gamma(2k+q+1)}{k!\,\Gamma(k+q)} G_k(q, q, x) = \frac{\Gamma(2n+q+1)}{n!\,\Gamma(n+q)} G_n(q+1, q, x),$$

or, since, $P_n^{(0,0)}(x) = \frac{(2n)!}{(n!)^2} G_n\left(1, 1, \frac{1+x}{2}\right) = P_n(x)$ and $P_n^{(1,1)}(x) = \frac{2}{n+1} \frac{d}{dx} P_{n+1}(x)$, where $P_n(x)$ is the LEGENDRE polynomial of order n, we find by substituting $p=q=1$ and $p=q=2$ in (1.3) respectively

$$(1.4)\qquad \sum_{k=0}^{n} (2k+1) P_k(x) = \frac{(2n+1)!}{(n!)^2} G_n\left(2, 1, \frac{1+x}{2}\right),$$

and

$$(1.5)\qquad \sum_{k=0}^{n} \frac{(2k+1)!}{(k!)^2} G_k\left(2, 2, \frac{1+x}{2}\right) = \frac{d}{dx} P_{n+1}(x).$$

The last two formulas are directly involved in determining the transfer functions of monotonic magnitude filters.

Substituting in (1.1) $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, $\lambda = 1$, $P_n^{(-1/2,-1/2)}(x) = \frac{(1/2)_n}{n!} T_n(x)$, where $T_n(x)$ is the CHEBYSHEV polynomial of the first kind, and collecting the product terms, we find

$$(1.6)\qquad \sum_{k=0}^{n} T_k(x) = \frac{(2n)!!}{2(2n-1)!!} P_n^{(1/2,-1/2)}(x) + \frac{1}{2}.$$

Similarly, for $\alpha = \beta + 1$, and, since $P_k^{(\beta,\beta)}(x) = \frac{(\beta+1)_k}{(2\beta+1)_k} C_k^{(\beta+1/2)}(x)$ where $C_k^{(\beta+1/2)}(x)$ is the GEGENBAUER polynomial, we obtain

$$(1.7)\qquad \sum_{k=0}^{n} (2k+2\beta+1) C_k^{(\beta+1/2)}(x) = \frac{(2\beta+1)_{n+1}}{(\beta+1)_n} P_n^{(\beta+1,\beta)}(x) - (2\beta+1).$$

Substituting $\beta = \frac{1}{2}$ and using the relation $C_n^{(1)}(x) = U_n(x)$, where $U_n(x)$ is the CHEBYSHEV polynomial of the second kind, (1.7) reduces to

$$(1.8)\qquad \sum_{k=0}^{n} (k+1) U_k(x) = \frac{(n+2)!}{2(3/2)_n} P_n^{(3/2,1/2)}(x).$$

2. In this part the following formulae will be used (see [5], p. 276):

$$(2.1)\qquad (2n+\lambda+1)(1+x) P_n^{(a,b+1)}(x) = 2(n+b+1) P_n^{(a,b)}(x) + 2(n+1) P_{n+1}^{(a,b)}(x),$$

$$(2.2)\qquad (2n+\lambda-1) P_n^{(a-1,b)}(x) = (n+\lambda-1) P_n^{(a,b)}(x) - (n+b) P_{n-1}^{(a,b)}(x),$$

$$(2.3)\qquad P_n^{(a,b-1)}(x) - P_n^{(a-1,b)}(x) = P_{n-1}^{(a,b)}(x),$$

$$(2.4)\qquad 2\frac{d}{dx} P_n^{(a,b)}(x) = (n+\lambda) P_{n-1}^{(a+1,b+1)}(x),$$

where $\lambda = a+b+1$.

From (2.4) and (2.3) we obtain

$$2(1+x)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=(1+x)(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{2.5}$$
$$=(1+x)(n+\alpha+\beta+1)\left(P_n^{(\alpha+1,\beta)}(x)-P_n^{(\alpha,\beta+1)}(x)\right).$$

Substituting $a=\alpha+1$ and $b=\beta-1$ in (2.1) we have

$$(2n+\alpha+\beta+2)\ (1+x)P_n^{(\alpha+1,\beta)}(x)=2(n+\beta)P_n^{(\alpha+1,\beta-1)}(x) \tag{2.6}$$
$$+2(n+1)P_{n+1}^{(\alpha+1,\beta-1)}(x),$$

while for $a=\alpha$ and $b=\beta$, (2.1) becomes

$$(2n+\alpha+\beta+2)(1+x)P_n^{(\alpha,\beta+1)}(x)=2(n+\beta+1)P_n^{(\alpha,\beta)}(x) \tag{2.7}$$
$$+2(n+1)P_{n+1}^{(\alpha,\beta)}(x).$$

By use of (2.1) and (2.3), the relation (2.5) can be written in the following form

$$2(1+x)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+2}\big(2(n+\beta)P_n^{(\alpha+1,\beta-1)}(x) \tag{2.8}$$
$$-2(n+\beta+1)P_n^{(\alpha,\beta)}(x)+2(n+1)P_n^{(\alpha+1,\beta)}(x)\big).$$

Now, substituting $a=\alpha+1$, $b=\beta$ in (2.2) and taking into account (2.3) we find

$$(n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)}(x)=(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) \tag{2.9}$$
$$+(n+\beta)\ \left(P_n^{(\alpha+1,\beta-1)}(x)-P_n^{(\alpha,\beta)}(x)\right).$$

Eliminating $P_n^{(\alpha+1,\beta)}(x)$ from (2.8) and (2.9), and rearranging the terms we deduce

$$(1+x)\frac{d}{dx}P_n^{(\alpha,\beta)}(x)=(n+\beta)P_n^{(\alpha+1,\beta-1)}(x)-\beta P_n^{(\alpha,\beta)}(x). \tag{2.10}$$

The special case of (2.10), obtained for $\alpha=0$, $\beta=1$,

$$(1+x)\frac{d}{dx}P_n^{(0,1)}(x)=(n+1)P_n^{(1,0)}(x)-P_n^{(0,1)}(x) \tag{2.11}$$

derives its origin from an interrelation between different subclasses of monotonic magnitude filters.

REFERENCES

1. A. Papoulis: *Optimum filters with monotonic response.* Proc. IRE **46** (1958), 606—609.
2. A. Papoulis: *On monotonic response of filters.* Proc. IRE **47** (1959), 331—333.
3. M. Fukada: *Optimum filters of even orders with monotonic response.* IRE Trans. Circuit Theory, **CT—6** (1959), 277—281.
4. P. H. Halpern: *Optimum monotonic low-pass filters.* IEEE Trans. Circuit Theory, **CT—16** (1969), 240—242.
5. Y. L. Luke: *The Special Functions and Their Approximations*, Vol. I. New York 1969.