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419. TWO ELEMENTARY INEQUALITIES*

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This Note is a supplement to the collection of inequalities appearing in Part 3 of the book: D. S. MITRINOVIĆ *Analytic Inequalities*. Berlin-Heidelberg-New York, 1970.

1. *If* $a_k \geqq 0,\ n \leqq a \leqq n+1,\ n=0,\ 1,\ \ldots,$ *the implication*

$$\sum_{k=1}^{n+1} \frac{1}{1+a_k} \geqq a \Rightarrow \prod_{k=1}^{n+1} a_k \leqq \left(\frac{n+1-a}{a}\right)^{n+1} \tag{1.1}$$

is valid.

Proof. Let $\sum x_1 \cdots x_k$ $(k=1, \ldots, n+1)$ denote the sum of all products having the form $x_{i_1} \cdots x_{i_k}$ with $1 \leqq i_1 < \cdots < i_k \leqq n+1$, and let $b^{n+1} = a_1 \cdots a_{n+1}$.

The left-hand side of the implication (1.1) is equivalent to

$$\sum (1+a_1) \cdots (1+a_n) \geqq a(1+a_1) \cdots (1+a_{n+1}),$$

i.e.

$$n+1+\sum_{k=1}^{n} (n-k+1) \sum a_1 \cdots a_k \geqq a + \sum_{k=1}^{n+1} a \cdot \sum a_1 \cdots a_k;$$

therefrom, upon a short arrangement and using the relationship between means of various sequences, it follows, one after the other:

$$\begin{aligned} n+1-a &\geqq \sum_{k=1}^{n+1} (a-n+k-1) \sum a_1 \cdots a_k \\ &\geqq \sum_{k=1}^{n+1} (a-n+k-1) \binom{n+1}{k} (a_1 \cdots a_{n+1})^{\frac{k}{n+1}} = \sum_{k=1}^{n+1} (a-n+k-1) \binom{n+1}{k} b^k \\ &= (a-n-1) \sum_{k=1}^{n+1} \binom{n+1}{k} b^k + b \left(\sum_{k=1}^{n+1} \binom{n+1}{k} x^k \right)'_{x=b} \\ &= (a-n-1)\left((1+b)^{n+1}-1\right) + b\left((1+x)^{n+1}-1\right)'_{x=b} \\ &= (1+b)^n (a-n-1+ab) + (n+1-a), \end{aligned}$$

i. e.

$$(1+b)^n (a-n-1+ab) \leqq 0 \Rightarrow b \leqq \frac{n+1-a}{a},$$

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and finally

$$a_1 \cdots a_{n+1} = b^{n+1} \leqq \left(\frac{n+1-a}{a}\right)^{n+1},$$

which was to be proved.

EXAMPLE. If we put $a_k = \mathrm{tg}^2 x_k$ $(k=1, \ldots, n+1)$ in (1.1), for $n \leqq a \leqq n+1$ we get the implication

$$\sum_{k=1}^{n+1} \cos^2 x_k \geqq a \Rightarrow \prod_{k=1}^{n+1} \mathrm{tg}^2 x_k \leqq \left(\frac{n+1-a}{a}\right)^{n+1}.$$

REMARK 1. In the book [1] implication 3.2.44 reads

$$(1.2) \qquad \sum_{k=1}^{n+1} \frac{1}{1+a_k} \geqq n \Rightarrow \prod_{k=1}^{n+1} \frac{1}{a_k} \geqq n^{n+1} \qquad (a_k > 0;\ k=1, \ldots, n+1).$$

Implication (1.1) for $a=n$ reduces to (1.2). A rather complicated proof for (1.2) was given in Elemente der Mathematik **14** (1959), 132 by C. BINDSCHEDLER.

2. *For $a_i \in [k, +\infty)$ $(k=1, 2, \ldots)$ we have*

$$(2.1) \qquad \frac{k}{k+1} \leqq \frac{\left(\prod_{i=1}^{n} a_i\right) \Big/ \left(\sum_{i=1}^{n} a_i\right)}{\left(\prod_{i=1}^{n} [a_i]\right) \Big/ \left(\sum_{i=1}^{n} [a_i]\right)} \leqq \left(\frac{k+1}{k}\right)^n \qquad (n=1, 2, \ldots)$$

where $[a]$ denotes the integral part of the real number a.

Proof. Inequality (2.1) may be written in the form

$$(2.2) \qquad \frac{k}{k+1} \leqq \prod_{i=1}^{n} \frac{a_i}{[a_i]} \cdot \frac{\sum_{i=1}^{n} [a_i]}{\sum_{i=1}^{n} a_i} \leqq \left(\frac{k+1}{k}\right)^n,$$

in which it will be proved.

Since the sequence $\left(\frac{n+1}{n}\right)_{n=1,2,\ldots}$ decreases for any $x \geqq k$ the inequality $\frac{x}{[x]} \leqq \frac{k+1}{k}$ holds, so that the right-hand inequality in (2.2) follows directly because $[a_i] \leqq a_i$ $(i=1, \ldots, n)$.

On the other hand, from $\frac{a_i}{[a_i]} \leqq \frac{k+1}{k}$ follows $[a_i] \geqq \frac{k}{k+1} a_i$ $(i=1, \ldots, n)$. or, after the addition,

$$\sum_{i=1}^{n} [a_i] \geqq \frac{k}{k+1} \sum_{i=1}^{n} a_i$$

Since it is obvious that $\frac{a_i}{[a_i]} \geqq 1$ $(i=1, \ldots, n)$, we conclude that the left-hand inequality in (2.2) is valid, too. Thereby inequality (2.2) as well as, inequality (2.1) is proved.

REMARK 2. For $a_i \in [n, +\infty)$ we have

$$\frac{n}{n+1} \leqq \frac{\left(\prod_{i=1}^{n} a_i\right) \Big/ \left(\sum_{i=1}^{n} a_i\right)}{\left(\prod_{i=1}^{n} [a_i]\right) \Big/ \left(\sum_{i=1}^{n} [a_i]\right)} < e.$$

REMARK 3. The following inequality

$$1 \leqq \frac{\left(\prod_{i=1}^{n} a_i\right) \left(\sum_{i=1}^{n} a_i\right)}{\left(\prod_{i=1}^{n} [a_i]\right) \left(\sum_{i=1}^{n} [a_i]\right)} \leqq \left(\frac{k+1}{k}\right)^{n+1}$$

holds under same conditons under which (2.1) is valid.

COMMENT OF B. CRSTICI RELEVANT TO IMPLICATION (1.1)

P. HENRICI (Elem. Math. **11** (1959), 112; see also D. S. MITRINOVIĆ: *Nejednakosti*, Beograd 1965, p. 165) proved the following implication

$$0 \leqq a_1, \ldots, a_{n+1} \leqq 1 \Rightarrow \sum_{k=1}^{n+1} \frac{1}{1+a_k} \leqq \frac{n+1}{1+\left(\prod_{k=1}^{n+1} a_k\right)^{1/(n+1)}}.$$

Therefrom it follows that if a is any positive number for which

$$\sum_{k=1}^{n+1} \frac{1}{1+a_k} \geqq a, \text{ then } \prod_{k=1}^{n+1} a_k \leqq \left(\frac{n+1-a}{a}\right)^{n+1},$$

i.e. KALAJDŽIĆ's result is obtained for any possible positive a, but with $0 \leqq a_1, \ldots, a_{n+1} \leqq 1$. KALAJDŽIĆ obtained that implication for any $a_1, \ldots, a_{n+1} \geqq 0$ but with a restriction for a $(a \geqq n)$.

Now, the following problem arises. P. HENRICI proved that if $a_1, \ldots, a_{n+1} \geqq 1$, then

$$\sum_{k=1}^{n+1} \frac{1}{1+a_k} \geqq \frac{n+1}{1+\left(\prod_{k=1}^{n+1} a_k\right)^{1/(n+1)}}.$$

Therefrom, it follows that the implication

$$\sum_{k=1}^{n+1} \frac{1}{1+a_k} \leqq a \Rightarrow \prod_{k=1}^{n+1} a_k \geqq \left(\frac{n+1-a}{a}\right)^{n+1} \qquad (a_1, \ldots, a_{n+1} \geqq 1)$$

is valid for $a_k \geqq 1$.

In the light of KALAJDŽIĆ's paper, it would be interesting to see which additional condition should be satisfied by a, in order to get that implication for any $a_k \geqq 0$ $(k=1, \ldots, n+1)$.

AUTHOR'S COMMENT

It is not difficult to see that for the quoted proof of the implication (1.1) the conditton $a \geqq n$ is essential. However, the condition $a \leqq n+1$ is only formally quoted because $\sum \frac{1}{1+a_k}$ cannot be greater than $n+1$.

Further, if we put $\sum_{k=1}^{n+1} \frac{1}{1+a_k} = b \geqq a$ in (1.1) we get

$$\prod_{k=1}^{n+1} a_k \leqq \left(\frac{n+1-b}{b}\right)^{n+1} \leqq \left(\frac{n+1-a}{a}\right)^{n+1},$$

where from HENRICI's inequality follows

$$\sum_{k=1}^{n+1} \frac{1}{1+a_k} \leqq \frac{n+1}{1+\left(\prod_{k=1}^{n+1} a_k\right)^{1/(n+1)}}$$

for any $a_1, \ldots, a_{n+1} \geqq 0$ for which $\sum_{k=1}^{n+1} \frac{1}{1+a_k} \geqq n$ (we encounter condition $0 \leqq a_1, \ldots, a_{n+1} \leqq 1$ in HENRICI's paper).

As far as the implication

$$(1.1)' \qquad \sum_{k=1}^{n+1} \frac{1}{1+a_k} \leqq a \Rightarrow \prod_{k=1}^{n+1} a_k \geqq \left(\frac{n+1-a}{a}\right)^{n+1}$$

is concerned, from the quoted proof for the implication (1.1) it follows that the implication (1.1)′ will hold for any $a_k \geqq 0$ $(k=1, \ldots, n+1)$ if $0 < a \leqq 1$ (namely, then $a-n+k-1 \leqq 0$ for $k=1, \ldots, n$).

If we put $\sum_{k=1}^{n+1} \frac{1}{1+a_k} = b \leqq a$ in (1.1)′, then we get

$$\prod_{k=1}^{n+1} a_k \geqq \left(\frac{n+1-b}{b}\right)^{n+1} \geqq \left(\frac{n+1-a}{a}\right)^{n+1},$$

and therefrom HENRICI's inequality follows

$$\sum_{k=1}^{n+1} \frac{1}{1+a_k} \geqq \frac{n+1}{1+\left(\prod_{k=1}^{n+1} a_k\right)^{1/(n+1)}}$$

for any $a_1, \ldots, a_{n+1} \geqq 0$ with $\sum_{k=1}^{n+1} \frac{1}{1+a_k} \leqq 1$ (we encounter condition $a_1, \ldots, a_{n+1} \geqq 1$ in HENRICI's paper).

REFERENCES

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2. D. S. MITRINOVIĆ — P. M. VASIĆ: *Généralisation d'une inégalité de Henrici.* These Publications № **210** — № **228** (1969), 35—38.
3. D. S. MITRINOVIĆ. *Elementarne Nierówności.* Warszawa, 1972, p. 141.