PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 412 — № 460 (1973)

416. ON GENERALIZED BOURGET'S FUNCTION*

Radovan R. Janić and *Žarko Mitrović*

1. In the book [1], BOURGET's function $J_{k,m}(x)$ is defined by means of

$$J_{k,m}(x)=\frac{1}{\pi}\int_0^{\pi}(2\cos\theta)^m\cos(k\theta-x\sin\theta)\,d\theta,$$

where k is an integer and m a natural number.

In paper [2] K. GOROWARA proved that the generating function for $J_{k,m}(x)$ is given by

$$(t+t^{-1})^m\exp\frac{x}{2}(t-t^{-1})=\sum_{k=-\infty}^{+\infty}J_{k,m}(x)\,t^k. \tag{1.1}$$

BESSEL's function of the first kind of the entire index k of n arguments $J_k(x_1, \ldots, x_n)$ was defined by M. AKIMOV [3] as a coefficient of t^k in the development

$$\exp\sum_{r=1}^{n}\frac{x_r}{2}(t^r-t^{-r})=\sum_{k=-\infty}^{+\infty}J_k(x_1,\ldots,x_n)\,t^k. \tag{1.2}$$

2. In this paper we shall define BOURGET's function of the entire index k of n arguments $J_{k,m}(x_1, \ldots, x_n)$ and we shall derive some formulas for it.

BOURGET's function $J_{k,m}(x_1, \ldots, x_n)$ is a coefficient of t^k in the following development

$$G(t)=\prod_{r=1}^{n}(t^r+t^{-r})^m\exp\sum_{r=1}^{n}\frac{x_r}{2}(t^r-t^{-r})=\sum_{k=-\infty}^{+\infty}J_{k,m}(x_1,\ldots,x_n)\,t^k. \tag{2.1}$$

Starting from (2.1) and (1.1) we have

$$\prod_{r=1}^{n-1}(t^r+t^{-r})^m\exp\sum_{r=1}^{n-1}\frac{x_r}{2}(t^r-t^{-r})=\sum_{p=-\infty}^{+\infty}J_{p,m}(x_1,\ldots,x_{n-1})\,t^p, \tag{2.2}$$

$$(t^n+t^{-n})^m\exp\frac{x_n}{2}(t^n-t^{-n})=\sum_{q=-\infty}^{+\infty}J_{q,m}(x_n)\,t^{nq}. \tag{2.3}$$

---

* Presented March 16, 1973 by S. K. CHATTERJEA.

Multiplying (2.2) and (2.3) term by term and comparing it with (2.1) we get

$$(2.4)\quad \sum_{k=-\infty}^{+\infty} J_{k,m}(x_1, \ldots, x_n)\, t^k = \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} J_{p,m}(x_1, \ldots, x_{n-1})\, J_{q,m}(x_n)\, t^{p+nq}.$$

If we put $p+nq=k$ in (2.4) and equate coefficients of t^k, we obtain

$$(2.5)\quad J_{k,m}(x_1, \ldots, x_n) = \sum_{q=-\infty}^{+\infty} J_{k-nq,m}(x_1, \ldots, x_{n-1})\, J_{q,m}(x_n).$$

Iterating the above procedure we arrive at the following formula

$$(2.6)\quad J_{k,m}(x_1, \ldots, x_n) = \sum_{q_2=-\infty}^{+\infty} \cdots \sum_{q_n=-\infty}^{+\infty} J_{k-\lambda,m}(x_1)\, J_{q_2,m}(x_2) \cdots J_{q_n,m}(x_n),$$

where $\lambda = \sum_{r=2}^{n} r q_r$.

If we substitute in (2.1) t by t^{-1}, we have

$$G(t^{-1}) = \prod_{r=1}^{n} (t^r + t^{-r})^m \exp\left(- \sum_{r=1}^{n} \frac{x_r}{2} (t^r - t^{-r}) \right),$$

wherefrom follows

$$(2.7)\quad G(t) = G(t^{-1}) \exp 2 \sum_{r=1}^{n} \frac{x_r}{2} (t^r - t^{-r}).$$

According to (1.2)

$$\exp 2 \sum_{r=1}^{n} \frac{x_r}{2} (t^r - t^{-r}) = \sum_{k=-\infty}^{+\infty} J_k(2x_1, \ldots, 2x_n)\, t^k,$$

so that (2.7) becomes

$$(2.8)\quad \sum_{k=-\infty}^{+\infty} J_{k,m}(x_1, \ldots, x_n)\, t^k = \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} J_{-p,m}(x_1, \ldots, x_n)\, J_q(2x_1, \ldots, 2x_n)\, t^{p+q}.$$

Putting $p+q=k$ in (2.8) and equating coefficients of t^k in (2.8), we obtain

$$J_{k,m}(x_1, \ldots, x_n) = \sum_{p=-\infty}^{+\infty} J_{-p,m}(x_1, \ldots, x_n)\, J_{k-p}(2x_1, \ldots, 2x_n),$$

i.e., substituting p by $-p$,

$$(2.9)\quad J_{k,m}(x_1, \ldots, x_n) = \sum_{p=-\infty}^{+\infty} J_{p,m}(x_1, \ldots, x_n)\, J_{k+p}(2x_1, \ldots, 2x_n).$$

3. If x_m is substituted by x_m+y_m $(m=1, \ldots, n)$ in (2.1), we obtain

$$(3.1)\quad \prod_{r=1}^{n} (t^r + t^{-r})^m \exp \sum_{r=1}^{n} \frac{x_r+y_r}{2} (t^r - t^{-r}) = \sum_{k=-\infty}^{+\infty} J_{k,m}(x_1+y_1, \ldots, x_n+y_n)\, t^k.$$

On the other hand we have

$$(3.2)\qquad \prod_{r=1}^{n}(t^r+t^{-r})^m \exp\sum_{r=1}^{n}\frac{x_r+y_r}{2}(t^r-t^{-r})$$

$$=\prod_{r=1}^{n}(t^r+t^{-r})^m \exp\sum_{r=1}^{n}\frac{x_r}{2}(t^r-t^{-r})\exp\sum_{r=1}^{n}\frac{y_r}{2}(t^r-t^{-r})$$

$$=\sum_{p=-\infty}^{+\infty}J_{p,m}(x_1,\ldots,x_n)\,t^p\sum_{q=-\infty}^{+\infty}J_q(y_1,\ldots,y_n)\,t^q.$$

Comparing (3.1) and (3.2) we find

$$(3.3)\qquad \sum_{k=-\infty}^{+\infty}J_{k,m}(x_1+y_1,\ldots,x_n+y_n)\,t^k$$

$$=\sum_{p=-\infty}^{+\infty}\sum_{q=-\infty}^{+\infty}J_{p,m}(x_1,\ldots,x_n)\,J_q(y_1,\ldots,y_n)\,t^{p+q}.$$

If we put $p+q=k$ in (3.3) and equate the coefficients of t^k, we obtain

$$(3.4)\qquad J_{k,m}(x_1+y_1,\ldots,x_n+y_n)=\sum_{p=-\infty}^{+\infty}J_{p,m}(x_1,\ldots,x_n)\,J_{k-p}(y_1,\ldots,y_n).$$

We now start with

$$(3.5)\qquad \prod_{r=1}^{n}(t^r+t^{-r})^{m+s}\exp\sum_{r=1}^{n}\frac{x_r+y_r}{2}(t^r-t^{-r})$$

$$=\prod_{r=1}^{n}(t^r+t^{-r})^m\exp\sum_{r=1}^{n}\frac{x_r}{2}(t^r-t^{-r})\prod_{r=1}^{n}(t^r+t^{-r})^s\exp\sum_{r=1}^{n}\frac{y_r}{2}(t^r-t^{-r})$$

$$=\sum_{p=-\infty}^{+\infty}J_{p,m}(x_1,\ldots,x_n)\,t^p\sum_{q=-\infty}^{+\infty}J_{q,s}(y_1,\ldots,y_n)\,t^q.$$

If m is substituted by $m+s$ in (3.1) and the obtained result compared with (3.5), in the same way as above, we conclude that

$$(3.6)\qquad J_{k,m+s}(x_1+y_1,\ldots,x_n+y_n)=\sum_{p=-\infty}^{+\infty}J_{p,m}(x_1,\ldots,x_n)\,J_{k-p,s}(y_1,\ldots,y_n).$$

4. If (2.1) is differentiated with respect to $x_i\,(i=1,\ldots,n)$ we obtain

$$\prod_{r=1}^{n}(t^r+t^{-r})^m\exp\sum_{r=1}^{n}\frac{x_r}{2}(t^r-t^{-r})\,\frac{t^i-t^{-i}}{2}=\sum_{k=-\infty}^{+\infty}\frac{\partial}{\partial x_i}J_{k,m}(x_1,\ldots,x_n)\,t^k,$$

which, upon rearrangement, yields

$$(4.1)\qquad \sum_{k=-\infty}^{+\infty}J_{k,m}(x_1,\ldots,x_n)\,t^{k+i}-\sum_{k=-\infty}^{+\infty}J_{k,m}(x_1,\ldots,x_n)\,t^{k-i}$$

$$=2\sum_{k=-\infty}^{+\infty}\frac{\partial}{\partial x_i}J_{k,m}(x_1,\ldots,x_n)\,t^k.$$

Comparing the coefficients of t^k in (4.1) we have

$$J_{k-i,m}(x_1, \dots, x_n) - J_{k+i,m}(x_1, \dots, x_n) = \frac{\partial}{\partial x_i} J_{k,m}(x_1, \dots, x_n). \tag{4.2}$$

If we differentiate (4.2) with respect to x_j $(j=1, \dots, n)$, we obtain

$$4\frac{\partial^2 J_{k,m}}{\partial x_i \partial x_j} = J_{k-i-j,m} - J_{k-i+j,m} - J_{k+i-j,m} + J_{k+i+j,m}, \tag{4.3}$$

where $J_{k,m} = J_{k,m}(x_1, \dots, x_n)$.

Using induction we may prove that

$$2^p \frac{\partial^p J_{k,m}}{\partial x_r{}^p} = J_{k-rp,m} - \binom{p}{1} J_{k-r(p-2),m} + \cdots + (-1)^p J_{k+rp,m}. \tag{4.4}$$

5. Starting from (2.1) we have

$$\prod_{r=1}^{n} (t^r + t^{-r})^{m+1} \exp \sum_{r=1}^{n} \frac{x^r}{2}(t^r - t^{-r}) = \sum_{k=-\infty}^{+\infty} J_{k,m+1}(x_1, \dots, x_n) t^k.$$

The left-hand side of this equality may be written in the form

$$\prod_{r=1}^{n} (t^r + t^{-r}) \sum_{p=-\infty}^{+\infty} J_{p,m}(x_1, \dots, x_n)\, t^p,$$

so that we have

$$\prod_{r=1}^{n} (t^r + t^{-r}) \sum_{p=-\infty}^{+\infty} J_{p,m}(x_1, \dots, x_n)\, t^p = \sum_{k=-\infty}^{+\infty} J_{k,m+1}(x_1, \dots, x_n)\, t^k. \tag{5.1}$$

Since

$$\prod_{r=1}^{n} (t^r + t^{-r}) = \prod_{r=1}^{n} \frac{1+t^{2r}}{t^r} = t^{-s}\left(1 + t^2 + t^4 + 2\sum_{i=3}^{s-3} t^{2i} + t^{2s-4} + t^{2s-2} + t^{2s}\right)$$

$$= t^{-s} + t^{2-s} + t^{4-s} + 2\sum_{i=3}^{s-3} t^{2i-s} + t^{s-4} + t^{s-2} + t^s,$$

where $s = 1 + 2 + \cdots + n$, according to (5.1), we have

$$\sum_{k=-\infty}^{+\infty} J_{k,m+1}(x_1, \dots, x_n)\, t^k \tag{5.2}$$

$$= \sum_{p=-\infty}^{+\infty} \left(t^{-s} + t^{2-s} + t^{4-s} + 2\sum_{i=3}^{s-3} t^{2i-s} + t^{s-4} + t^{s-2} + t^s\right) t^p J_{p,m}(x_1, \dots, x_n).$$

Equating coefficients of t^k in (5.2) we obtain

$$J_{k,m+1} = J_{k+s,m} + J_{k+s-2,m} + J_{k+s-4,m} + 2\sum_{i=3}^{s-3} J_{k+s-2i,m} \tag{5.3}$$

$$+ J_{k-s+4,m} + J_{k-s+2,m} + J_{k-s,m},$$

where $2s = n(n+1)$.

6. The coefficient $I_{k,m}(x_1, \ldots, x_n)$ of t^k in the development

$$\prod_{r=1}^{n} (t^r + t^{-r})^m \exp \sum_{r=1}^{n} \frac{x_r}{2}(t^r + t^{-r}) = \sum_{k=-\infty}^{+\infty} I_{k,m}(x_1, \ldots, x_n)\, t^k$$

will be called modified BOURGET'S function of n arguments.

For modified BOURGET'S function $I_{k,m}(x_1, \ldots, x_n)$ formulas similar to (2.5), (2.6), (2.9), (3.4), (3.6), (4.2), (4.4) and (5.3) may be deduced, some of them being the same.

We may also define the functions $J^1_{k,m}(x_1, \ldots, x_n)$ and $I^1_{k,m}(x_1, \ldots, x_n)$ as coefficients of t^k in the developments

$$\prod_{r=1}^{n} (t^r - t^{-r})^m \exp \sum_{r=1}^{n} \frac{x_r}{2}(t^r - t^{-r}) = \sum_{k=-\infty}^{+\infty} J^1_{k,m}(x_1, \ldots, x_n)\, t^k,$$

$$\prod_{r=1}^{n} (t^r - t^{-r})^m \exp \sum_{r=1}^{n} \frac{x_r}{2}(t^r + t^{-r}) = \sum_{k=-\infty}^{+\infty} I^1_{k,m}(x_1, \ldots, x_n)\, t^k.$$

We may also consider similar functions generated by

$$\prod_{r=1}^{n} (t^r + t^{-r})^m \exp \sum_{r=1}^{n} \frac{x_r}{2}(t^r + (-1)^r t^{-r});$$

$$\prod_{r=1}^{n} (t^r + (-1)^r t^{-r})^m \exp \sum_{r=1}^{n} \frac{x_r}{2}(t^r + t^{-r});$$

$$\prod_{r=1}^{n} (t^r + (-1)^r t^{-r})^m \exp \sum_{r=1}^{n} \frac{x_r}{2}(t^r + (-1)^r t^{-r}),$$

etc.

Results similar to the previous ones may be deduced for these functions.

* * *

The authors wish to thank Professor S. K. CHATTERJEA for useful comments and suggestions.

REFERENCES

1. E. T. WHITTAKER and G. N. WATSON: *A course of modern analysis.* Cambridge 1962.
2. K. K. GOROWARA: *On Bourget's function $J_{n,k}(z)$.* Ganita **22** (1971), 21—26.
3. M. J. AKIMOFF: *Transcendentes de Fourier-Bessel à plusieurs variables.* C. R. Acad. Sci. Paris **165** (1917), 23—25.

Katedra za matematiku
Elektrotehnički fakultet
11000 Beograd, Jugoslavija

Elektronski fakultet
18000 Niš, Jugoslavija