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415. A PROBLEM OF A. OPPENHEIM*

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0. In [4] A. OPPENHEIM asked the following question:

Suppose $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *are two sets of positive numbers. If the arithmetic mean of the* b_i *is at least equal to that of the* a_i, *when can we say that the geometric mean of the* b_i *is at least equal to that of the* a_i *and that equality will require equality in some order of the* a_i *and* b_i?

When $n=2$ the only non-trivial situation, if we assume, as we may, that

$$a_1 \leqq a_2, \quad b_1 \leqq b_2$$

is when

$$a_1 \leqq b_1 \leqq b_2 \leqq a_2.$$

Then clearly $b_1+b_2 \geqq a_1+a_2$ implies $b_1 b_2 \geqq a_1 a_2$ as is seen from the simple identity,

$$b_1 b_2 - a_1 a_2 = (b_1 - a_1)(a_2 - b_1) + b_1 (b_1 + b_2 - a_1 - a_2);$$

equality occurs only when $a_1 = b_1$ and $a_2 = b_2$.

This case also follows from the case $n=3$ (by taking $b_3 = a_3$) which was completely solved in [4]; in a later paper [5], the same author showed that the hypothesis sufficient to answer his question gave in fact a stronger result. The problem of $n>3$ was posed in [4] where it was pointed out that the obvious extension was false. An extension to general n was given a little later by E. K. GODUNOVA and V. I. LEVIN [2]; (compare Theorems 1 and 2 below). Another extension to arbitrary n was given by P. M. VASIĆ in [7].

Before stating these results let us introduce some standard notation. Let $(w)=(w_1, \ldots, w_n)$ denote an n-tple of positive numbers and write $W_k = \sum_{i=1}^{k} w_i$

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$(1 \leq k \leq n)$. If $(a) = (a_1, \ldots, a_n)$ is another such n-tple write

$$M_n^{[r]}(a;\, w) = \left(\frac{1}{W_n} \sum_{i=1}^{n} a_i^r w_i\right)^{\frac{1}{r}} \qquad (0 < |r| < +\infty),$$

$$= \left(\prod_{i=1}^{n} a_i^{w_i}\right)^{\frac{1}{W_n}} \qquad (r = 0),$$

$$= \min\,(a_1, \ldots, a_n) \quad (r = -\infty),$$

$$= \max\,(a_1, \ldots, a_n) \quad (r = +\infty).$$

More generally if Φ is a strictly monotonic function

$$M_n^{[\Phi]}(a;\, w) = \Phi^{-1}\left(\frac{1}{W_n} \sum_{i=1}^{n} \Phi(a_i)\, w_i\right).$$

If $r = 1, 0, -1$ there are special notations: $M_n^{[1]}(a;\, w) = A_n(a;\, w)$; $M_n^{[0]}(a;\, w) = G_n(a;\, w)$; $M_n^{[-1]}(a;\, w) = H_n(a;\, w)$. If $w_1 = \ldots = w_n$ we will just write $M_n^{[r]}(a)$, $A_n(a)$ etc.

1. The main result of Oppenheim is

Theorem 1. *Let (a), (b) be two triples of positive numbers satisfying*
(h) the elements of (b) lie between the greatest and least element of (a).

If $0 \leqq \alpha \leqq \frac{2}{3}$ *and*

$$A_3(b) \geqq A_3(a),$$

then

$$G_3(b) \geqq \left\{\frac{A_3(b)}{A_3(a)}\right\}^{\alpha} G_3(a); \tag{1}$$

in particular

$$G_3(b) \geqq G_3(a). \tag{2}$$

Equality occurs in (1) *or* (2) *if and only if the* (a) *is a rearrangement of* (b).

The proof of (2) is given in [4] and that of (1) is the main purpose of [5]. Clearly inequality (1) for any particular α $\left(0 \leqq \alpha \leqq \frac{2}{3}\right)$ implies, given the hypothesis, the same inequality for smaller α. In particular $\alpha = \frac{2}{3}$ is the strongest inequality; in [5] Oppenheim shows that if $\alpha > 2/3$ the inequality no longer holds in general.

Since, in Theorem 1, we can obviously assume (a), and (b) to be monotonic increasing, the hypothesis (h) is equivalent to $a_1 \leqq b_1$ and $b_3 \leqq a_3$. The problem posed by Oppenheim in [4] was to find a suitable generalisation of (h): this was done in [2]. The Godunova—Levin extension of Theorem 1 is

Theorem 2. *Let $n>2$ and (a), (b) be two n-tples of positive numbers satisfying*

$$(H)\qquad \begin{cases} 0<a_1\leqq \dots \leqq a_n, \\ 0<b_1\leqq \dots \leqq b_n F, \\ a_i\leqq b_i \quad (1\leqq i \leqq n-m,\ 1\leqq m\leqq n), \\ a_i\geqq b_i \quad (n-m+2\leqq i\leqq n). \end{cases}$$

(If $m=1$ the last condition is understood to be vacuous; if $m=n$ the next to last condition is taken to be vacuous).

*If (p) is another n-*tple *of positive numbers,* $0\leqq\alpha\leqq 1-\frac{P_{n-m}}{P_n}$, *and*

$$A_n(b;\,p)\geqq A_n(a;\,p)$$

then

$$(3)\qquad G_n(b;\,p)\geqq\left\{\frac{A_n(b;\,p)}{A_n(a;\,p)}\right\}^{\alpha} G_n(a;\,p);$$

in particular

$$G_n(b;\,p)\geqq G_n(a;\,p).$$

If $n=3$, $m=2$ then hypothesis (*H*) is equivalent to (*h*) as noted above; if in addition $p_1=p_2=p_3$ Theorem 2 reduces to Theorem 1. Although not stated in this form in [2], it is proved there as a simple corollary of a more general result that is given below (Theorem 7).

2. A natural extension of OPPENHEIM's problem quoted above is to replace arithmetic and geometric means by more general means. OPPENHEIM did not ask this question although some of his results give partial answers to this more general problem; more surprising is the fact that the question was not raised in [2] as their general theorem can be used to answer part of this problem.

More precisely we are asking: if it is known that the s-th power means of (*a*), (*b*) are in a certain order when can we deduce that the same order holds between their t-th power means?

In section 1 we dealt with the case $s=1$, $t=0$; in [4, 5] the case $s=0$, $t=1$ as well as some other cases were given.

Theorem 3. *Let (a), (b) be two triples of positive numbers satisfying (h).*

(i) If $r>0$ and

$$M_3^{[r]}(b)\geqq M_3^{[r]}(a)$$

then

$$G_3(b)\geqq G_3(a).$$

(ii) If

$$G_3(a)\geqq G_3(b)$$

and if $r>0$ then

$$M_3^{[r]}(a)\geqq M_3^{[r]}(b).$$

(iii) If

$$A_3(b) \geqq A_3(a)$$

then

$$H_3(a) \geqq \left(\frac{G_3(a)}{G_3(b)}\right)^3 H_3(b).$$

(iv) If

$$H_3(b) \geqq \left(\frac{G_3(b)}{G_3(a)}\right)^3 H_3(a)$$

then

$$A_3(a) \leqq A_3(b).$$

Equality occurs only when (a) *is a rearrangement of* (b).

Parts *(iii)* and *(iv)* are weaker than the „natural" answer to the above question. Parts *(i)*, *(ii)* are easy extensions of the case $r=1$; in **[5]** it is shown that whereas *(i)* can be sharpened to (1), no such extension is possible for *(ii)*.

3. Theorem 4. *Let* $\Phi: R_+ \to R$ *be increasing and concave and suppose* (a), (b) *are two n-tples of positive numbers satisfying* (H). *Let* (p) *be another n-tple of positive numbers, if*

$$A_n(b;\, p) \geqq A_n(a;\, p) \tag{4}$$

then

$$\sum_{i=1}^{n} \Phi(b_i)\, p_i \geqq \sum_{i=1}^{n} \Phi(a_i)\, p_i; \tag{5}$$

if Φ *is strictly increasing, then*

$$M_n^{[\Phi]}(b;\, p) \geqq M_n^{[\Phi]}(a;\, p). \tag{6}$$

If Φ' *is positive at one point at least then equality occurs in* (5) *if and only if*

$$a_i = b_i \; (1 \leqq i \leqq n).$$

Proof. Let $c_i = \lambda b_i + (1-\lambda) a_i \quad (1 \leqq i \leqq n)$; then (c) can replace either (a) or (b) in (H); in particular (c) is increasing.

Note that since Φ is concave Φ' exists except on a countable set and is decreasing; further since Φ is increasing Φ' is non-negative.

Define

$$F(\lambda) = \sum_{i=1}^{n} p_i \Phi(c_i).$$

Then it is necessary to prove that $F(1) \geqq F(0)$, since F' exists except on a countable set it is sufficient to prove $F'(\lambda) \geqq 0$ whenever it exists.

If $1 \leqq i \leqq n-m$ then $b_i \geqq a_i$ and, from the above remark, $\Phi'(c_i) \geqq \Phi'(c_{n-m+1})$, hence,

$$\sum_{i=1}^{n-m} p_i (b_i - a_i)\, \Phi'(c_i) \geqq \sum_{i=1}^{n-m} p_i (b_i - a_i)\, \Phi'(c_{n-m+1});$$

similarly if $n-m+1 \leqq i \leqq n$, $b_i \leqq a_i$, so

$$\sum_{i=n-m+1}^{n} p_i (b_i - a_i) \Phi'(c_i) \geqq \sum_{i=n-m+1}^{n} p_i (b_i - a_i) \Phi'(c_{n-m+1}).$$

From these remarks the following inequalities are easily checked

$$F'(\lambda) = \sum_{i=1}^{n} p_i (b_i - a_i) \Phi'(c_i)$$

$$\geqq \sum_{i=1}^{n} p_i (b_i - a_i) \Phi'(c_{n-m+1}) \geqq 0;$$

which completes the proof of (5).

Inequality (6) and the cases of equality are immediate.

By inspection of the above proof similar results can be obtained using different hypotheses on Φ; in summary they are as follows.

(A) If Φ is convex and increasing the inequality (4) is reversed then inequalities (5) and (6) are reversed.

(B) If Φ is convex and decreasing then inequality (5) is reversed but (6) holds.

Consider the case $\Phi(x) = x^r$:

(i) if $0 < r \leqq 1$ the hypotheses of Theorem 4 hold:

(ii) if $r \geqq 1$ then hypotheses in (A) hold;

(iii) if $r < 0$ then hypotheses in (B) hold.

If $\Phi(x) = \log x$ the hypotheses of Theorem 4 hold. If $\Phi(x) = e^{\lambda x}$, then hypothesis (A) holds if $\lambda > 0$ but if $\lambda < 0$ then (B) applies.

These are sufficient to completely solve the problem posed above.

Corollary 5. *Let* (a), (b) *be two n-tples of positive numbers satisfying* (H) *and* (p) *another n-tple of positive numbers. If* $-\infty < s < +\infty$ *then*

(i) *if*

$$M_n^{[s]}(b; p) \geqq M_n^{[s]}(a; p)$$

and if $t < s$, *then*

(7) $$M_n^{[t]}(b; p) \geqq M_n^{[t]}(a; p);$$

(ii) *if*

$$M_n^{[s]}(a; p) \geqq M_n^{[s]}(b; p)$$

and if $t > s$, *then*

(8) $$M_n^{[t]}(a; p) \geqq M_n^{[t]}(b; p).$$

Equality occurs in (7) *or* (8) *if and only if* $a_i = b_i$ $(1 \leqq i \leqq n)$.

Corollary 5 has a very interesting implication.

Corollary 6. *Suppose* (a), (b) *are two n-tples of positive numbers, one not being a rearrangement of the other; suppose further that* (p) *is another n-tple of positive numbers. If*

$$M_n^{[-\infty]}(a;\, p) < M_n^{[-\infty]}(b;\, p), \qquad M_n^{[+\infty]}(a;\, p) > M_n^{[+\infty]}(b;\, p) \tag{9}$$

then if (a), (b) *satisfy* (H) *there is a unique* s $(-\infty < s < +\infty)$ *such that*

(*i*) *if* $t < s$ *then* $M_n^{[t]}(a;\, p) < M_n^{[t]}(b;\, p)$;

(*ii*) *if* $t > s$ *then* $M_n^{[t]}(a;\, p) > M_n^{[t]}(b;\, p)$;

(*iii*) $M_n^{[s]}(a;\, p) = M_n^{[s]}(b;\, p)$.

Proof. Immediate.

This poses the following interesting question: suppose instead of (9) we assume that for some u, v, $(-\infty \leqq u < v \leqq +\infty)$,

$$M_n^{[u]}(a;\, p) < M_n^{[u]}(b;\, p) \qquad M_n^{[v]}(a;\, p) > M_n^{[v]}(b;\, p)$$

does an s $(u < s < v)$, exist with the properties similar to those in Corollary 6?

4. A part of corollary 5 (*i*) follows from the result of GODUNOVA and LEVIN, [2], although they did not state this explicitly. In fact a stronger result holds — an analogue of (3); since as we have pointed above OPPENHEIM showed that such an analogue does not hold in general, [5], all of Corollary 5 cannot be obtained in this way. The main theorem in [2] is

Theorem 7. *Suppose* (a), (b) *are two n-tples of positive numbers satisfying* (H) *and let* (p) *be another n-tple of positive numbers. Let* $\Phi : R_+ \to R$ *be such that* (*i*) Φ *is concave,* (*ii*) Φ *is increasing,* (*iii*) $x\Phi'(x)$ *is increasing. If*

$$A_n(b;\, p) \geqq A_n(a;\, p) \tag{10}$$

then if $0 \leqq \alpha \leqq 1 - \dfrac{P_{n-m}}{P_n}$

$$A_n(\Phi(a);\, p) - \alpha\Phi(A_n(a;\, p)) \leqq A_n(\Phi(b);\, p) - \alpha\Phi(A_n(b;\, p)) \tag{11}$$

($\Phi(a)$ *denotes the n-tple* $(\Phi(a_1), \ldots, \Phi(a_n))$).

Inspection of the proof in [2] gives the following.

(A') If Φ is convex and increasing wih $x\Phi'(x)$ increasing and if inequality (10) is reversed then inequality (11) is reversed.

(B') If Φ is convex and decreasing and $x\Phi'(x)$ is decreasing then inequality (11) is reversed.

Corollary 8. *Let* (a), (b) *be two n-tples of positive numbers satisfying* (H) *and let* (p) *be another n-tple of positive numbers. If* $0 \leqq \alpha \leqq 1 - \dfrac{P_{n-m}}{P_n}$ *and* $-\infty < s < +\infty$ *then*

(*i*) *if*

$$M_n^{[s]}(b;\, p) \geqq M_n^{[s]}(a;\, p) \tag{12}$$

and if $t<s$,

$$\{(M_n^{[t]}(b;\,p))^t - \alpha(M_n^{[s]}(b;\,p))^t\}^{1/t} \tag{13}$$
$$\geqq \{(M_n^{[s]}(a;\,p))^t - \alpha(M_n^{[s]}(a;\,p))^t\}^{1/t} \qquad (t\neq 0),$$

$$G_n(b;\,p) \geqq \left\{\frac{M_n^{[s]}(b;\,p)}{M_n^{[s]}(a;\,p)}\right\}^\alpha G_n(a;\,p) \qquad (t=0), \tag{14}$$

(*ii*) *if we assume the reverse of* (12) *and if* $t>s$ *the reverse of* (13) *or* (14) *holds. Equality occurs in* (13) *or* (14) *if and only if* $a_i=b_i$ $(1\leqq i\leqq n)$.

Proof. Immediate from Theorem 7 taking $\Phi(x)$ variously equal to x^r, $\log x$ or e^x.

It was pointed out in [5] that if in (*ii*) we take $s=0$, $t=1$ the inequality analogous to the reverse of (14) does not hold in general (i.e. with G_n replaced by A_n and $M_n^{[s]}$ by G_n); the correct inequality is now seen to be the reverse of (13) (with $s=0$, $t=1$).

5. Theorem 4 can be generalized by replacing the conditions (*H*) by (*F*) below, based on a classical condition due to Hardy, Littlewood and Pólya, [**6**, p. 162—166].

Theorem 9. *Let* (*a*), (*b*), (*p*) *be n-tples of real numbers with* (*a*) *and* (*b*) *satisfying*

$$a_1\leqq\cdots\leqq a_n,\quad b_1\leqq\cdots\leqq b_n,$$
$$\sum_{i=1}^{k} a_i p_i \leqq \sum_{i=1}^{k} b_i p_i \qquad (1\leqq k\leqq n-m,\quad 1\leqq m\leqq n), \tag{F}$$
$$\sum_{i=k}^{n} a_i p_i \geqq \sum_{i=k}^{n} b_i p_i \qquad (n-m+2\leqq k\leqq n).$$

(*If* $m=1$ *the last condition is understood to be vacuous; if* $m=n$ *the next to last condition is taken to be vacuous.*)

(*i*) *If* $\Phi: R\to R$ *is concave and*

$$\sum_{i=1}^{n} b_i p_i = \sum_{i=1}^{n} a_i p_i \tag{15}$$

then (5) *holds.*

(*ii*) *If* $\Phi: R\to R$ *is concave and increasing and* (4) *holds, i.e.*

$$\sum_{i=1}^{n} b_i p_i \geqq \sum_{i=1}^{n} a_i p_i$$

then (5) *holds; further equality holds under the same conditions as in Theorem* 4.

Proof. Let us define (*c*) and F as in the proof of Theorem 4. Then, taking note of the remarks in that proof, it is sufficient to prove that $F'\geqq 0$.

Let us write

$$B_k = \sum_{i=1}^{k} b_i p_i, \quad A_k = \sum_{i=1}^{k} a_i p_i \qquad (1 \leq k \leq n);$$

$$A_0 = B_0 = 0;$$

$$B_k' = B_n - B_k, \quad A_k' = A_n - A_k \qquad (1 \leq k \leq n).$$

Then

$$\begin{aligned} F'(\lambda) &= \sum_{i=1}^{n} p_i (b_i - a_i) \Phi'(c_i) \\ &= \sum_{i=1}^{n-m} (B_i - A_i) \{\Phi'(c_i) - \Phi'(c_{i+1})\} \\ &\quad + \sum_{i=n-m+1}^{n-1} (A_i' - B_i') \{\Phi'(c_i) - \Phi'(c_{i+1})\} \\ &\quad + (B_n - A_n) \Phi'(c_{n-m+1}). \end{aligned}$$

With the hypothesis (F) and the hypotheses of Φ this identity gives $F'(\lambda) \geqq 0$ as had to be proved. The case of equality is immediate.

REMARKS. (i) If the weights (p) are non-negative then clearly (H) implies (F). More however is true: (H) and (4) imply (F), with $m=1$, and (4); quite obviously $a_i \leqq b_i$ $(1 \leqq i \leqq n-m)$ implies $A_k \leqq B_k$ $(1 \leqq k \leqq n-m)$ and if $k > n-m$ then $A_k = A_n - A_k' \leqq B_n - B_k' = B_k$, by (H) and (4). The converse is false as is seen by taking $n=3$, $p_1 = p_2 = p_3 = 1$, $a_1 = 1$, $a_2 = a_3 = 5$, $b_1 = b_2 = 3$, $b_3 = 6$.

(ii) In Theorem 9 (ii) if Φ is strictly increasing clearly (6) holds.

(iii) As with Theorem 4, inspection of the above proof shows that similar results can be obtained with different hypotheses. Some of these are as follows.

(α) Assumptions: (a) and (b) decreasing; rest of (F) the same; Φ convex; (15) holds. Conclusion: (5) holds.

(β) Assumptions: as in (α) except that Φ is convex and increasing and (4) holds. Same conclusion as (α).

(γ) Assumptions: (a), (b) increasing; rest of (F) reversed; (4) reversed; Φ convex and decreasing. Conclusion is that then (5) holds.

(δ) Assumptions: (a), (b) increasing; rest of (F) reversed; (4) reversed; Φ concave and increasing. Conclusion is that the reverse of (5) holds.

(iv) Theorem 9 (α) is a slight extension of a result of L. FUCH's [6, p. 165]; FUCH's result is the case $m=1$.

The possibility of replacing the hypothesis (H) by (F) in Corollary 5, and hence Corollary 6, is not immediate. Corollary 5 is proved by applying a suitable particular case of Theorem 4 to the sequences (a^s) and (b^s); this is possible since if (a) and (b) satisfy (H) so do (a^s) and (b^s); the case $s \leqq 0$ needs slight extra modifications. Whether this is so for hypothesis (F) is the subject of the next corollary.

Corollary 10. (i) *If hypothesis (F) with $m=1$ holds for (a) and (b) and if Φ is concave increasing then it holds for $(\Phi(a))$ and $(\Phi(b))$.*

(ii) *If hypothesis (F) with $m=n$ holds for (a) and (b) and if Φ is convex increasing then it holds for $(\Phi(a))$ and $(\Phi(b))$.*

(*iii*) *If hypothesis* (*F*) *with* $1<m<n$ *holds for* (*a*) *and* (*b*) *and if* Φ *is linear increasing then it holds for* $(\Phi(a))$ *and* $(\Phi(b))$.

Proof. As the proof will show it is sufficient to consider (*iii*). Since Φ is increasing

$$\Phi(a_1) \leqq \cdots \leqq \Phi(a_n), \qquad \Phi(b_1) \leqq \cdots \leqq \Phi(b_n).$$

Now apply Theorem 9 (*ii*) to $(a_1, \ldots, a_k)$, and $(b_1, \ldots, b_k)$ $(1 \leqq k \leqq n-m)$ to get that

$$\sum_{i=1}^{k} \Phi(a_i) p_i \leqq \sum_{i=1}^{k} \Phi(b_i) p_i \qquad (1 \leqq k \leqq n-m).$$

Finally apply Theorem 9 (β) to the sequences $(b_n, b_{n-1}, \ldots, b_k)$ and $(a_n, a_{n-1}, \ldots, a_k)$ $(n-m+2 \leqq k \leqq n)$ to get

$$\sum_{i=k}^{n} \Phi(a_i) p_i \geqq \sum_{i=k}^{n} \Phi(b_i) p_i \qquad (n-m+2 \leqq k \leqq n).$$

REMARKS. (*i*) It follows from this that in general Corollary 5 will only extend partially depending on which hypothesis (F) is chosen.

Given three n-tples (*a*), (*b*) and (*p*) with (*a*) and (*b*) decreasing, i.e..

$$a_1 \geqq \cdots \geqq a_n, \qquad b_1 \geqq \cdots \geqq b_n$$

then let us say for a given Φ, strictly monotonic, that (*b*) Φ-dominates (*a*) with weight (*p*) if

$$M_k^{[\Phi]}(b; p) \geqq M_k^{[\Phi]}(a; p) \qquad (1 \leqq k \leqq n);$$

in particular we will say that (*b*) *s*-dominates (*a*) with weight (*p*) if $\Phi(x) = x^s$ $(s \neq 0)$, $\Phi(x) = \log x$ $(s=0)$; finally if $s=1$ we will just say (*b*) dominates (*a*) with weight (*p*).

REMARKS. (*i*) For all Φ and (*p*) this defines an order relation on the set of decreasing n-tples.

(*ii*) If $s=1$, $p_1 = \cdots = p_n$ then this order reduces to one introduced by HARDY, LITTLEWOOD and PÓLYA [6, p. 163].

(*iii*) Theorem 9 (β) shows that if (*b*) dominates (*a*) with weight (*p*) then (*b*) Φ-dominates (*a*) with weight (*p*) for all convex strictly increasing Φ.

Corollary 11. *If* (*b*) *s-dominates* (*a*) *with weight* (*p*) $(-\infty < s < +\infty)$ *then* (*b*) *t-dominates* (*a*) *with weight* (*p*) *for all* $t>s$.

Proof. (*i*) Suppose $s>0$; then the hypothesis is equivalent to saying (b^s) dominates (a^s) and so the result follows from the preceding remark (*iii*) with $\Phi(x) = x^r$ $\left(r = \frac{t}{s},\ t>s\right)$.

(*ii*) Suppose $s<0$; then the hypothesis is equivalent to saying that the increasing sequences (a^s), (b^s) satisfy the reverse of the remaining inequalites of (*F*), $m=1$, and the reverse of (4). Then if $t>0$ and $\Phi(x) = x^r$ $\left(r = \frac{t}{s}\right)$ the result follows from Theorem 9 (γ); if $t \leqq 0$, $\Phi(x) = x^r \left(r = \frac{t}{s},\ t \neq 0\right)$, $\Phi(x) = \log x$, $t = 0$ the result follows from Theorem 9 (δ).

(*iii*) A similar argument covers the case $s = 0$ using $\Phi(x) = e^{tx}$ $(t > 0)$.

6. It would be of interest to know if similar results hold for the symmetric and counter-harmonic means, [3, p. 79].

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