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414. ČEBYŠEV INEQUALITY FOR CONVEX SETS*

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Theorem 1. *If* $A=(A_1, \ldots, A_n)$ *and* $B=(B_1, \ldots, B_n)$ *are two real sequences such that*

$$A_1 \geqq \cdots \geqq A_n \quad and \quad B_1 \geqq \cdots \geqq B_n,$$

or

$$A_1 \leqq \cdots \leqq A_n \quad and \quad B_1 \leqq \cdots \leqq B_n,$$

then the following inequality holds

$$(1) \qquad T_n(A, B; P) \geqq T_{n-1}(A, B; P)$$

where

$$T_n(A, B; P) = \sum_{i=1}^{n} P_i \sum_{i=1}^{n} P_i A_i B_i - \sum_{i=1}^{n} P_i A_i \sum_{i=1}^{n} P_i B_i$$

and $P=(P_1, \ldots, P_n)$ *is a positive sequence.*

Equality in (1) *holds if and only if* $A_i = A_n \, (i \in I \subset \{1, \ldots, n\})$, $B_j = B_n \, (j \in \{1, \ldots, n\} \setminus I)$ *while* $A_i \, (i \in \{1, \ldots, n\} \setminus I)$ *and* $B_j \, (j \in I)$ *are arbitrary.*

Proof. The statement of the Theorem 1 follows immediately from the following identity

$$\begin{aligned} T_n(A, B; P) - T_{n-1}(A, B; P) &= \sum_{i=1}^{n} P_i \sum_{i=1}^{n} P_i A_i B_i - \sum_{i=1}^{n} P_i A_i \sum_{i=1}^{n} P_i B_i \\ &\quad - \sum_{i=1}^{n-1} P_i \sum_{i=1}^{n-1} P_i A_i B_i + \sum_{i=1}^{n-1} P_i A_i \sum_{i=1}^{n-1} P_i B_i \\ &= \left(P_n + \sum_{i=1}^{n-1} P_i\right) \left(P_n A_n B_n + \sum_{i=1}^{n-1} P_i A_i B_i\right) \\ &\quad - \left(P_n A_n + \sum_{i=1}^{n-1} P_i A_i\right) \left(P_n B_n + \sum_{i=1}^{n-1} P_i B_i\right) \\ &\quad - \sum_{i=1}^{n-1} P_i \sum_{i=1}^{n-1} P_i A_i B_i + \sum_{i=1}^{n-1} P_i A_i \sum_{i=1}^{n-1} P_i B_i \end{aligned}$$

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$$=P_n\left(\sum_{i=1}^{n-1} P_i A_i B_i-\sum_{i=1}^{n-1} P_i A_i B_n+\sum_{i=1}^{n-1} P_i A_n B_n-\sum_{i=1}^{n-1} P_i A_n B_i\right)$$

$$=P_n \sum_{i=1}^{n-1} P_i (A_i-A_n)(B_i-B_n).$$

REMARK 1. The proof of Theorem 1 is analogous to the proof of inequality 3.2.28 in [1]. Inequality 3.2.28 is a particular case of inequality (1) (for $P_1=\cdots=P_n=1$).

REMARK 2. Since $T_1(A, B; P)=0$ and

$$T_n(A, B; P) \geqq T_{n-1}(A, B; P) \geqq \cdots \geqq T_1(A, B; P)=0,$$

we have

$$\sum_{i=1}^{n} P_i \sum_{i=1}^{n} P_i A_i B_i-\sum_{i=1}^{n} P_i A_i \sum_{i=1}^{n} P_i B_i \geqq 0. \tag{2}$$

Inequality (2) is the well-known ČEBYŠEV inequality.

Theorem 2. *If* $a=(a_1, \ldots, a_n)$ *and* $b=(b_1, \ldots, b_n)$ *are two real sequences such that*

$$0=a_1 \leqq \cdots \leqq a_n, \qquad 0=b_1 \leqq \cdots \leqq b_n,$$

$$a_{i-1}-2a_i+a_{i+1} \geqq 0, \quad b_{i-1}-2b_i+b_{i+1} \geqq 0 \qquad (i=2, \ldots, n-1),$$

then the following inequality is valid

$$C_n(a, b; p) \geqq C_{n-1}(a, b; p), \tag{3}$$

where

$$C_n(a, b; p)=\left\{\sum_{i=1}^{n} p_i (i-1)\right\}^2 \sum_{i=1}^{n} p_i a_i b_i-\sum_{i=1}^{n} p_i (i-1)^2 \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i$$

and $p=(p_1, \ldots, p_n)$ *is a positive sequence.*

Equality in (3) *holds if and only if* $a_1=\cdots=a_n$, $b_1=\cdots=b_n$.

Proof. On the basis of

$$a_{i-1}-2a_i+a_{i+1} \geqq 0 \qquad (i=2, \ldots, k-1; \quad 3 \leqq k \leqq n)$$

we have

$$(i-1)(a_{i-1}-2a_i+a_{i+1}) \geqq 0 \qquad (i=2, \ldots, k-1; \quad 3 \leqq k \leqq n)$$

and

$$\sum_{i=2}^{k-1} (i-1)(a_{i-1}-2a_i+a_{i+1})$$

$$=a_1+(k-3)a_{k-1}-2(k-2)a_{k-1}+(k-1)a_k \geqq 0,$$

i.e.,

$$\frac{a_k}{k-1} \geqq \frac{a_{k-1}}{k-2} \qquad (3 \leqq k \leqq n).$$

Similarly,

$$\frac{b_k}{k-1} \geqq \frac{b_{k-1}}{k-2} \qquad (3 \leqq k \leqq n).$$

Therefore, the sequences $\left(\frac{a_r}{r-1}\right)_{r=2,\ldots,n}$ and $\left(\frac{b_r}{r-1}\right)_{r=2,\ldots,n}$ are nondecreasing and nonnegative and consequently $\left(\frac{a_r b_r}{(r-1)^2}\right)_{r=2,\ldots,n}$ is nondecreasing.

If we put

$$P_i=p_i(i-1),\quad A_i=i-1,\quad B_i=\frac{a_i b_i}{(i-1)^2}\qquad (i=2,\ldots,n)$$

and $P_1=A_1=B_1=0$, we have from (1)

$$\sum_{i=2}^{n} p_i(i-1)\sum_{i=2}^{n} p_i a_i b_i-\sum_{i=2}^{n} p_i(i-1)^2\sum_{i=2}^{n} p_i\frac{a_i b_i}{i-1} \tag{4}$$
$$\geqq\sum_{i=2}^{n-1} p_i(i-1)\sum_{i=2}^{n-1} p_i a_i b_i-\sum_{i=2}^{n-1} p_i(i-1)^2\sum_{i=2}^{n-1} p_i\frac{a_i b_i}{i-1},$$

since P_i, A_i, B_i satisfy the conditions of Theorem 1.

Similarly, for

$$P_i=p_i(i-1),\quad A_i=\frac{a_i}{i-1},\quad B_i=\frac{b_i}{i-1}\qquad (i=2,\ldots,n)$$

and $P_1=A_1=B_1=0$ we obtain from (1)

$$\sum_{i=2}^{n} p_i(i-1)\sum_{i=2}^{n} p_i\frac{a_i b_i}{i-1}-\sum_{i=2}^{n} p_i a_i\sum_{i=2}^{n} p_i b_i \tag{5}$$
$$\geqq\sum_{i=2}^{n-1} p_i(i-1)\sum_{i=2}^{n-1} p_i\frac{a_i b_i}{i-1}-\sum_{i=2}^{n-1} p_i a_i\sum_{i=2}^{n-1} p_i b_i.$$

By combining inequalities (4) and (5), we get

$$\left\{\sum_{i=2}^{n} p_i(i-1)\right\}^2\sum_{i=2}^{n} p_i a_i b_i-\sum_{i=2}^{n} p_i(i-1)^2\sum_{i=2}^{n} p_i a_i\sum_{i=2}^{n} p_i b_i \tag{6}$$
$$\geqq\sum_{i=2}^{n} p_i(i-1)\sum_{i=2}^{n-1} p_i(i-1)\sum_{i=2}^{n-1} p_i a_i b_i-\sum_{i=2}^{n} p_i(i-1)\sum_{i=2}^{n-1} p_i(i-1)^2\sum_{i=2}^{n-1} p_i\frac{a_i b_i}{i-1}$$
$$+\sum_{i=2}^{n} p_i(i-1)^2\sum_{i=2}^{n-1} p_i(i-1)\sum_{i=2}^{n-1} p_i\frac{a_i b_i}{i-1}-\sum_{i=2}^{n} p_i(i-1)^2\sum_{i=2}^{n-1} p_i a_i\sum_{i=2}^{n-1} p_i b_i$$
$$=\left\{p_n(n-1)+\sum_{i=2}^{n-1} p_i(i-1)\right\}\sum_{i=2}^{n-1} p_i(i-1)\sum_{i=2}^{n-1} p_i a_i b_i$$
$$-\left\{p_n(n-1)+\sum_{i=2}^{n-1} p_i(i-1)\right\}\sum_{i=2}^{n-1} p_i(i-1)^2\sum_{i=2}^{n-1} p_i\frac{a_i b_i}{i-1}$$
$$+\left\{p_n(n-1)^2+\sum_{i=2}^{n-1} p_i(i-1)^2\right\}\sum_{i=2}^{n-1} p_i(i-1)\sum_{i=2}^{n-1} p_i\frac{a_i b_i}{i-1}$$

$$-\left\{p_n(n-1)^2+\sum_{i=2}^{n-1}p_i(i-1)^2\right\}\sum_{i=2}^{n-1}p_ia_i\sum_{i=2}^{n-1}p_ib_i$$

$$=\left\{\sum_{i=2}^{n-1}p_i(i-1)\right\}^2\sum_{i=2}^{n-1}p_ia_ib_i-\sum_{i=2}^{n-1}p_i(i-1)^2\sum_{i=2}^{n-1}p_ia_i\sum_{i=2}^{n-1}p_ib_i$$

$$+p_n(n-1)^2\left\{\sum_{i=2}^{n-1}p_i(i-1)\sum_{i=2}^{n-1}p_i\frac{a_ib_i}{i-1}-\sum_{i=2}^{n-1}p_ia_i\sum_{i=2}^{n-1}p_ib_i\right\}$$

$$+p_n(n-1)\left\{\sum_{i=2}^{n-1}p_i(i-1)\sum_{i=2}^{n-1}p_ia_ib_i-\sum_{i=2}^{n-1}p_i(i-1)^2\sum_{i=2}^{n-1}p_i\frac{a_ib_i}{i-1}\right\}.$$

Since, according to the ČEBYŠEV inequality

$$\sum_{i=2}^{n-1}p_i(i-1)\sum_{i=2}^{n-1}p_i\frac{a_ib_i}{i-1}-\sum_{i=2}^{n-1}p_ia_i\sum_{i=2}^{n-1}p_ib_i\geqq 0$$

and

$$\sum_{i=2}^{n-1}p_i(i-1)\sum_{i=2}^{n-1}p_ia_ib_i-\sum_{i=2}^{n-1}p_i(i-1)^2\sum_{i=2}^{n-1}p_i\frac{a_ib_i}{i-1}\geqq 0,$$

it follows from (6)

$$\left\{\sum_{i=2}^{n}p_i(i-1)\right\}^2\sum_{i=2}^{n}p_ia_ib_i-\sum_{i=2}^{n}p_i(i-1)^2\sum_{i=2}^{n}p_ia_i\sum_{i=2}^{n}p_ib_i \tag{7}$$

$$\geqq\left\{\sum_{i=2}^{n-1}p_i(i-1)\right\}^2\sum_{i=2}^{n-1}p_ia_ib_i-\sum_{i=2}^{n-1}p_i(i-1)^2\sum_{i=2}^{n-1}p_ia_i\sum_{i=2}^{n-1}p_ib_i.$$

Since $a_1=b_1=0$, $C_2(a, b; p)=0$, from (7) we get

$$C_n(a, b; p)\geqq C_{n-1}(a, b; p)\geqq\cdots\geqq C_2(a, b; p)=0, \tag{8}$$

so that we can write

$$\left\{\sum_{i=1}^{n}p_i(i-1)\right\}^2\sum_{i=1}^{n}p_ia_ib_i\geqq\sum_{i=1}^{n}p_i(i-1)^2\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ib_i. \tag{9}$$

Since, using the ČEBYŠEV inequality, we have

$$\frac{\sum_{i=1}^{n}p_i(i-1)^2}{\left\{\sum_{i=1}^{n}p_i(i-1)\right\}^2}\geqq\frac{1}{\sum_{i=1}^{n}p_i},$$

it may be concluded that inequality (9) is sharper than the ČEBYŠEV inequality, provided that the conditions of convexity and nonnegativity for the sequences a and b are added.

REFERENCE

1. D. S. MITRINOVIĆ: *Analytic Inequalities*. Berlin—Heidelberg—New York 1970.