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413. ON SOME INEQUALITIES FOR CONVEX AND g-CONVEX FUNCTIONS*

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1. Let $g: I^2 \rightarrow I$ be a given function such that $g(x, y) > 0$ for $y > x$ $(x, y \in I)$. It is said that $f: I \rightarrow R$ is a convex function on I if

$$g(x_2, x_3) f(x_1) + g(x_3, x_1) f(x_2) + g(x_1, x_2) f(x_3) \geqq 0 \tag{1.1}$$

is valid for every $x_1, x_2, x_3 \in I$ $(x_1 < x_2 < x_3)$ (see [**1**]).

Of particular interest are the g-convex functions satisfying $g(x, y) + g(y, x) = 0$, because several properties of convex (in the ordinary sense) functions may be associated with them. Hence, in this paper only those functions g will be dealt with for which

$$g(x, y) + g(y, x) = 0 \ \big(\Rightarrow g(x, y) = h(x, y) - h(y, x)\big)$$

holds.

The inequality

$$\frac{f(x_1)}{g(x_1, x_3)\, g(x_1, x_2)} + \frac{f(x_2)}{g(x_2, x_1)\, g(x_2, x_3)} + \frac{f(x_3)}{g(x_3, x_1)\, g(x_3, x_2)} \geqq 0 \tag{1.2}$$

is equivalent to (1.1), irrespective of the way in which the numbers x_1, x_2, x_3 are arranged in (1.2) subject to the constraint $x_1 \neq x_2 \neq x_3 \neq x_1$.

Substituting

$$x_1 = x_i, \ x_2 = \sum_{k=0}^{n} p_k x_k, \ x_3 = x_0 \ (p_k > 0, \ x_i \in I)$$

in (1.2), we get

$$\frac{f(x_i)}{g(x_i, x_0)\, g\left(x_i, \sum_{k=0}^{n} p_k x_k\right)} + \frac{f\left(\sum_{k=0}^{n} p_k x_k\right)}{g\left(\sum_{k=0}^{n} p_k x_k, x_i\right) g\left(\sum_{k=0}^{n} p_k x_k, x_0\right)} + \frac{f(x_0)}{g(x_0, x_i)\, g\left(x_0, \sum_{k=0}^{n} p_k x_k\right)} \geqq 0 \qquad (i = 1, \ldots, n). \tag{1.3}$$

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If

$$(1.4) \qquad g(x_i, x_0)\, g\left(x_i, \sum_{k=0}^{n} p_k x_k\right) < 0 \qquad (i=1, \ldots, n)$$

we obtain from (1.3)

$$(1.5) \quad p_i f(x_i) \leq \frac{p_i g(x_i, x_0)}{g\left(\sum_{k=0}^{n} p_k x_k, x_0\right)} f\left(\sum_{k=0}^{n} p_k x_k\right) + \frac{p_i g\left(x_i, \sum_{k=0}^{n} p_k x_k\right)}{g\left(x_0, \sum_{k=0}^{n} p_k x_k\right)} f(x_0) \quad (i=1, \ldots, n).$$

By summing the inequalities (1.5) we find

$$(1.6) \quad \sum_{i=1}^{n} p_i f(x_i) \leq \frac{\sum_{i=1}^{n} p_i g(x_i, x_0)}{g\left(\sum_{k=0}^{n} p_k x_k, x_0\right)} f\left(\sum_{k=0}^{n} p_k x_k\right) + \frac{\sum_{i=1}^{n} p_i g\left(x_i, \sum_{k=0}^{n} p_k x_k\right)}{g\left(x_0, \sum_{k=0}^{n} p_k x_k\right)} f(x_0).$$

Since $g(x_0, x_0) = 0$, for

$$(1.7) \qquad g\left(x_0, \sum_{k=0}^{n} p_k x_k\right) \neq 0$$

we have

$$p_0 f(x_0) = \frac{p_0 g(x_0, x_0)}{g\left(\sum_{k=0}^{n} p_k x_k, x_0\right)} f\left(\sum_{k=0}^{n} p_k x_k\right) + \frac{p_0 g\left(x_0, \sum_{k=0}^{n} p_k x_k\right)}{g\left(x_0, \sum_{k=0}^{n} p_k x_k\right)} f(x_0),$$

which, when added to (1.6), yields

$$(1.8) \quad \sum_{i=0}^{n} p_i f(x_i) \leq \frac{\sum_{i=0}^{n} p_i g(x_i, x_0)}{g\left(\sum_{k=0}^{n} p_k x_k, x_0\right)} f\left(\sum_{k=0}^{n} p_k x_k\right) + \frac{\sum_{i=0}^{n} p_i g\left(x_i, \sum_{k=0}^{n} p_k x_k\right)}{g\left(x_0, \sum_{k=0}^{n} p_k x_k\right)} f(x_0).$$

Therefore, we have proved that:

Theorem 1. *If the conditions* (1.4) *and* (1.7) *are valid, then for every g-convex function f inequality* (1.8) *holds.*

If we take $g(x, y) = y - x$, then g-convexity transforms into the ordinary convexity, and (1.4) becomes

$$(1.9) \qquad (x_i - x_0)\left(\sum_{k=0}^{n} p_k x_k - x_i\right) > 0 \qquad (i=1, \ldots, n).$$

In this case the inequality (1.8) reads:

$$(1.10)\quad \sum_{i=0}^{n} p_i f(x_i) \leqq \frac{\sum_{i=0}^{n} p_i x_i - x_0 \sum_{i=0}^{n} p_i}{\sum_{i=0}^{n} p_i x_i - x_0} f\left(\sum_{i=0}^{n} p_i x_i\right) - \frac{1-\sum_{i=0}^{n} p_i}{\sum_{i=0}^{n} p_i x_i - x_0} \sum_{i=0}^{n} p_i x_i f(x_0).$$

F. Giaccardi [1] proved the inequality (1.10) under the assumptions that f is a convex function and

$$(1.11)\qquad 0 \leqq x_0 < x_i < \sum_{i=0}^{n} p_i x_i \qquad (i=0, 1, \ldots, n).$$

Condition (1.9) is weaker than (1.11).

2. Let us write

$$(2.1)\qquad P_n(p, x; f) = f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) - \left(1 - \sum_{i=1}^{n} p_i\right) f(0).$$

If $x_0 = 0$, the inequality (1.10) yields

$$(2.2)\qquad P_n(p, x; f) \geqq 0$$

provided that f is a convex function,

$$(2.3)\qquad x_i\left(\sum_{k=1}^{n} p_k x_k - x_i\right) > 0 \qquad (i=1, \ldots, n)$$

and

$$(2.4)\qquad \sum_{k=1}^{n} p_k x_k \neq 0.$$

Evidently, the condition (2.4) is implied by (2.3) and hence it can the disregarded.

The inequality (2.2) is a particular case of this result under the conditions that f is a convex functon, $x_1, \ldots, x_n > 0$, $p_1, \ldots, p_n \geqq 1$ (see [2]) and M. Petrović's inequality (see: [3]):

$$(2.5)\qquad f\left(\sum_{i=1}^{n} x_i\right) \geqq \sum_{i=1}^{n} f(x_i) - (1-n) f(0)$$

where $x_1, \ldots, x_n \geqq 0$ and f is a convex function.

From (2.1) we have

$$P_n(p, x; f) - P_{n-1}(p, x; f) = f\left(\sum_{i=1}^{n} p_i x_i\right) - f\left(\sum_{i=1}^{n-1} p_i x_i\right) - p_n f(x_n) + p_n f(0).$$

For $n=2$, from (2.2), (2.3), we obtain

$$(2.6)\qquad f(p_1 x_1 + p_2 x_2) - p_1 f(x_1) - p_2 f(x_2) - (1 - p_1 - p_2) f(0) \geqq 0$$

under the conditions

$$(2.7)\qquad \begin{aligned} x_1(p_1 x_1 + p_2 x_2 - x_1) &\geqq 0,\\ x_2(p_1 x_1 + p_2 x_2 - x_2) &\geqq 0. \end{aligned}$$

Substituting

$$x_1 \to x_n,\ x_2 \to \sum_{i=1}^{n-1} p_i x_i,\ p_1 \to p_n,\ p_2 \to 1,$$

from (2.6) we find

$$P_n(p, x; f) - P_{n-1}(p, x; f) \geqq 0 \tag{2.8}$$

provided that f is a convex function and

$$\begin{gathered} x_n\left(\sum_{i=1}^{n} p_i x_i - x_n\right) > 0, \\ \sum_{i=1}^{n-1} p_i x_i\left(\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n-1} p_i x_i\right) = p_n x_n \sum_{i=1}^{n-1} p_i x_i > 0. \end{gathered} \tag{2.9}$$

Therefore, the following theorem is valid:

Theorem 2. *If f is a convex function and if the conditions* (2.9) *are valid, then the inequality* (2.8) *holds.*

Remark. By iteration of the inequality (2.8), we arrive at

$$P_n(p, x; f) \geqq P_{n-1}(p, x; f) \geqq \cdots \geqq P_1(p, x; f) = 0,$$

where f is a convex function and

$$x_k\left(\sum_{i=1}^{k} p_i x_i - x_k\right) > 0, \quad p_k x_k \sum_{i=1}^{k-1} p_i x_i > 0 \qquad (k = 1, \ldots, n).$$

This result is an improvement of (2.2).

3. N. Levinson [5] proved the inequality

$$\frac{\sum_{k=1}^{n} p_k f(x_k)}{\sum_{k=1}^{n} p_k} - f\left(\frac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k}\right) \leqq \frac{\sum_{k=1}^{n} p_k f(2a - x_k)}{\sum_{k=1}^{n} p_k} - f\left(\frac{\sum_{k=1}^{n} p_k f(2a - x_k)}{\sum_{k=1}^{n} p_k}\right) \tag{3.1}$$

under the assumption that $f'''(t) > 0$ for $t \in (0, 2a)$, $0 < x_k \leqq a$, $p_k > 0$.

P. M. Vasić and R. R. Janić [6] proved, by taking into account the supplementary assumptions $p_k \geqq 1$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$, that the inequality

$$\begin{aligned} \sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} p_k f(2a - x_k) \geqq f\left(\sum_{k=1}^{n} p_k x_k\right) - f\left(2a - \sum_{k=1}^{n} p_k x_k\right) \\ - \left(1 - \sum_{k=1}^{n} p_k\right)(f(0) - f(2a)) \end{aligned} \tag{3.2}$$

holds.

Putting $n=2$ in (3.1) and substituting

$$x_1 \to \sum_{k=0}^{n} p_k x_k, \quad x_2 \to x_0, \quad p_1 \to 1, \quad p_2 \to \frac{\sum_{k=0}^{n} p_k x_k - x_i}{x_i - x_0} \qquad (i=1, \dots, n)$$

we find

$$(3.3) \qquad f(x_i) - f(2a - x_i) \geqq \frac{x_i - x_0}{\sum_{k=0}^{n} p_k x_k - x_0} f\Big(\sum_{k=0}^{n} p_k x_k\Big)$$

$$+ \frac{\sum_{k=0}^{n} p_k x_k - x_i}{\sum_{k=0}^{n} p_k x_k - x_0} \big(f(x_0) - f(2a - x_0)\big)$$

$$- \frac{x_i - x_0}{\sum_{k=0}^{n} p_k x_k - x_0} f\Big(2a - \sum_{k=0}^{n} p_k x_k\Big) \qquad (i=1, \dots, n).$$

Therefore, after multiplication by $p_i > 0$, summation and addition of $f(x_0) - f(2a - x_0) = f(x_0) - f(2a - x_0)$ to the corresponding sides of the resulting inequality, we get

$$(3.4) \qquad \sum_{i=0}^{n} p_i f(x_i) - \sum_{i=0}^{n} p_i f(2a - x_i)$$

$$\geqq \frac{\sum_{i=0}^{n} p_i x_i - x_0 \sum_{i=0}^{n} p_i}{\sum_{i=0}^{n} p_i x_i - x_0} \Big(f\Big(\sum_{i=0}^{n} p_i x_i\Big) - f\Big(2a - \sum_{i=0}^{n} p_i x_i\Big)\Big)$$

$$+ \frac{\sum_{i=0}^{n} p_i - 1}{\sum_{i=0}^{n} p_i x_i - x_0} \sum_{i=0}^{n} p_i x_i \big(f(x_0) - f(2a - x_0)\big)$$

subject to the conditions

$$(x_i - x_0)\Big(\sum_{i=0}^{n} p_i x_i - x_0\Big) > 0 \qquad (i=1, \dots, n)$$

and $f'''(t) > 0$ for $t \in (0, 2a)$ and $\sum_{i=0}^{n} p_i x_i \in [0, a]$.

Inequality (3.3) yields (3.2) for $x_0 = 0$.

4. Let f be a convex function of the second order for $x \in (0, 2a)$ and let $x_1, \ldots, x_4 \in [0, a]$. Then the inequality

$$(4.1)\qquad \frac{f(x_1)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}+\frac{f(x_2)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}+\frac{f(x_3)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}+\frac{f(x_4)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \geqq 0$$

holds (see [**4**], pp. 15—17).

Upon substitution $x_i \to 2a-x_i$ $(i=1, \ldots, 4)$, we obtain from (4.1)

$$(4.2)\qquad \frac{-f(2a-x_1)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}+\frac{-f(2a-x_2)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}+\frac{-f(2a-x_3)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)}+\frac{-f(2a-x_4)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \geqq 0.$$

Replacing $x_4 = a$ in (4.1) and (4.2) and adding the obtained inequalities, we get

$$\frac{\dfrac{f(x_1)-f(2a-x_1)}{x_1-a}}{(x_1-x_2)(x_1-x_3)}+\frac{\dfrac{f(x_2)-f(2a-x_2)}{x_2-a}}{(x_2-x_1)(x_2-x_3)}+\frac{\dfrac{f(x_3)-f(2a-x_3)}{x_3-a}}{(x_3-x_1)(x_3-x_2)} \geqq 0.$$

Therefore, the function

$$x \mapsto \frac{f(x)-f(2a-x)}{x-a}$$

is convex (order 1) on $[0, a]$ and from Jensen's inequality we get

$$(4.3)\qquad \sum_{k=1}^{n} p_k \frac{f(x_k)-f(a-x_k)}{x_k-a} \leqq \sum_{k=1}^{n} p_k \frac{f\left(\dfrac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k}\right)-f\left(2a-\dfrac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k}\right)}{\sum_{k=1}^{n} p_k x_k - a\sum_{k=1}^{n} p_k}.$$

The inequality (3.1) does not imply the validity of (4.3), and vice versa.

This stems from the fact that the convexity of $x \mapsto g(x)$ does not mean of necessity that $x \mapsto \dfrac{g(x)}{x-a}$ is also convex, and vice versa.

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