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388. OPERATIONAL REPRESENTATION FOR THE ULTRASPHERICAL POLYNOMIALS*

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1. Introduction

The writer has been continuing his work in the field of special functions of mathematical physics since the year 1954. An abstract of his doctoral dissertation submitted in the University of Jadavpur, India, (1964), was published on the Mat. Vesnik **[1, 2]** of Jugoslavia. His post-doctoral works consist in studying special functions by means of differential operators. Many properties of special functions can be readily derived by using the differential operators with which the functions are associated. In the discussion of such operational methods, it is interesting to mention that these special functions are related to the theory of continuous transformation groups and LIE groups and in the derivation of the properties of special functions, the classical analytical methods may be replaced by the powerful concepts of group theory. The object of this paper is to point out the following operational representation for the Ultraspherical (GEGENBAUER) polynomials:

$$\left(\frac{x}{\sqrt{x^2-1}}\right)^n P_n^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right)=\frac{1}{n!}(x^2-1)^\lambda(-\delta)_n(x^2-1)^{-\lambda} \tag{1.1}$$

where $\delta\equiv x\frac{d}{dx}$, $(a)_n\equiv a(a+1)\cdots(a+n-1)$.

This operational representation is found to be very convenient in deriving the generating relations:

$$\sum_{n=0}^{+\infty} P_n^\lambda(x)\,t^n=(1-2xt+t^2)^{-\lambda}, \tag{1.2}$$

$$\sum_{m=0}^{+\infty}\binom{m+n}{n}P_{m+n}^\lambda(x)\,t^m=(1-2xt+t^2)^{-\frac{n}{2}-\lambda}P_n^\lambda\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right). \tag{1.3}$$

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Here it may be of interest to mention that the generating relation (1.2) is contained in the following generating relation of the writer [3]:

$$(1.4) \qquad \sum_{n=0}^{+\infty} (t/\mu)^n P_n^\lambda\left(\mu x + \frac{1-\mu^2}{2\mu} t\right) = (1-2xt+t^2)^{-\lambda} \qquad (\mu \text{ being arbitrary})$$

which was derived from the theory of continuous transformations groups. The importance of this type of generating relation, viz. (1.4), lies in the fact that it yields the addition and multiplication formulas, in addition to the usual generating relation (1.2).

Furthermore it may be noted that the generating relations (1.2) and (1.3) may be derived by LIE Algebra methods [4]. Here it will be shown how the LIE Algebra methods enable one to derive a particular pair of generating relations, viz. (3.6) and (3.7), for the Ultraspherical polynomials, which are novel extensions of (1.3). To derive this particular pair of generating relations we have to consider two infinitesimal differential operators R and L, appearing in the recursion relation for the Ultraspherical polynomials such that these operators raise and lower the indices in the function concerned and then we can generate from them the finite operators $\exp(\alpha R)$ and $\exp(\alpha L)$. These infinitesimal operators R and L are therefore considered as generators of LIE Algebra. Any element of the corresponding LIE group operates on the function in two ways: (i) it shifts the argument of the function, (ii) it produces an infinite sum of functions with unchanged argument and with shifted indices. Thus equating these two results of the operation we obtain the desired properties. Now if R and L commute, i. e. $[R, L)] \equiv RL - LR = 0$, we have the simple „composition law" viz.

$$(1.5) \qquad \exp(\alpha R) \exp(\beta L) = \exp(\alpha R + \beta L).$$

But when $[R, L] \neq 0$, then the left-hand member of (1.5) differs from the right-hand member and depends upon the successive commutators of R and L. Evidently, then we have

$$(1.6) \qquad \exp(\alpha R) \exp(\beta L) \neq \exp(\beta L) \exp(\alpha R).$$

In such a case we may well apply the operators $\exp(\alpha R) \exp(\beta L)$ and $\exp(\beta L) \exp(\alpha R)$ successively on the function concerned in order to derive a pair of generating relations for the function.

2. Operational Representation

F. TRICOMI [5] gave the following differentiation formula for the Ultraspherical polynomials:

$$(2.1) \qquad P_n^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{(-1)^n (x^2-1)^{\lambda+\frac{1}{2}n}}{n!} D^n (x^2-1)^{-\lambda}.$$

Using (2.1) we derive an interesting operational representation for such polynomial:

$$(2.2) \qquad \left(\frac{x}{\sqrt{x^2-1}}\right)^n P_n^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{n!} (x^2-1)^\lambda (-\delta)_n (x^2-1)^{-\lambda},$$

where $\delta \equiv x D$, $(a)_n \equiv a(a+1) \cdots (a+n-1)$.

We shall now consider some applications of our operational formula (2.2). For our purpose we require the following results:

$$a^{\delta} f(x) = f(ax), \tag{2.3}$$

$$(1+t)^{-\delta-\alpha} f(x) = \sum_{n=0}^{+\infty} \frac{(-t)^n}{n!} (\delta+\alpha)_n f(x), \tag{2.4}$$

$$(1+t)^{-\delta-\alpha} = (1+t)^{-\alpha} (1+t)^{-\delta}. \tag{2.5}$$

Now we have

$$\sum_{n=0}^{+\infty} \left(\frac{x}{\sqrt{x^2-1}}\right)^n P_n^{\lambda}\left(\frac{x}{\sqrt{x^2-1}}\right) t^n = (x^2-1)^{\lambda} \sum_{n=0}^{+\infty} \frac{t^n}{n!} (-\delta)_n (x^2-1)^{-\lambda}$$
$$= (x^2-1)^{\lambda} (1-t)^{\delta} (x^2-1)^{-\lambda}$$
$$= (x^2-1)^{\lambda} \{(x(1-t))^2 - 1\}^{-\lambda}$$
$$= \left\{\frac{(x(1-t))^2-1}{x^2-1}\right\}^{-\lambda}.$$

Changing $x(x^2-1)^{-1/2}$ into x and t into tx^{-1}, we obtain the well-known generating relation (1.2). Our method of derivation does not seem to appear in any earlier investigation.

Next we obtain

$$\sum_{m=0}^{+\infty} \binom{m+n}{n} \left(\frac{x}{\sqrt{x^2-1}}\right)^{m+n} P_{m+n}^{\lambda}\left(\frac{x}{\sqrt{x^2-1}}\right) t^m$$
$$= (x^2-1)^{\lambda} \sum_{m=0}^{+\infty} \binom{m+n}{n} \frac{t^m}{(m+n)!} (-\delta)_{m+n} (x^2-1)^{-\lambda}$$
$$= (x^2-1)^{\lambda} \sum_{m=0}^{+\infty} \binom{m+n}{n} \frac{t^m}{(m+n)!} (-\delta+n)_m (-\delta)_n (x^2-1)^{-\lambda}$$
$$= (x^2-1)^{\lambda} \sum_{m=0}^{+\infty} \frac{t^m}{m!} (-\delta+n)_m (x^2-1)^{-\lambda} \left(\frac{x}{\sqrt{x^2-1}}\right)^n P_n^{\lambda}\left(\frac{x}{\sqrt{x^2-1}}\right)$$
$$= (x^2-1)^{\lambda} (1-t)^{\delta-n} (x^2-1)^{-\lambda} \left(\frac{x}{\sqrt{x^2-1}}\right)^n P_n^{\lambda}\left(\frac{x}{\sqrt{x^2-1}}\right)$$
$$= (x^2-1)^{\lambda} (1-t)^{-n} (1-t)^{\delta} (x^2-1)^{-\lambda} \left(\frac{x}{\sqrt{x^2-1}}\right)^n P_n^{\lambda}\left(\frac{x}{\sqrt{x^2-1}}\right)$$
$$= \left[\frac{(x(1-t))^2-1}{x^2-1}\right]^{-\lambda} \left(\frac{x}{\sqrt{(x(1-t))^2-1}}\right)^n P_n^{\lambda}\left(\frac{x(1-t)}{\sqrt{(x(1-t))^2-1}}\right).$$

Now putting $x/\sqrt{x^2-1} = y$, so that $x = y/\sqrt{y^2-1}$, we obtain

$$\sum_{m=0}^{+\infty} \binom{m+n}{m} P_{m+n}^{\lambda}(y)(yt)^m = (1-2ty^2+t^2y^2)^{-\frac{1}{2}n-\lambda} P_n^{\lambda}\left(\frac{y(1-t)}{\sqrt{1-2ty^2+t^2y^2}}\right),$$

whence we derive

$$\sum_{m=0}^{+\infty} \binom{m+n}{n} P_{m+n}^{\lambda}(x) t^m = (1-2xt+t^2)^{-\frac{1}{2}n-\lambda} P_n^{\lambda}\left(\frac{x-t}{\sqrt{(1-2xt+t^2)}}\right),$$

which is (1.3). It may be of interest to mention that we obtained a new bilateral generating function [6] for the Ultraspherical polynomials by such operational method.

3. A particular pair of generating relations

In [4] we notice that for the Ultaspherical polynomials defined by the differential equation

$$\left[\frac{d^2}{d\theta^2}+2\lambda\cot\theta\frac{d}{d\theta}+l(l+2\lambda)\right]G_l^\lambda(\cos\theta)=0, \tag{3.1}$$

we have

$$\begin{aligned} R\cdot F_l^\lambda(\theta, t)&=-[(l+1)(l+2\lambda)]^{\frac{1}{2}}F_{l+1}^\lambda(\theta, t),\\ L\cdot F_l^\lambda(\theta, t)&=-[l(l+2\lambda-1)]^{\frac{1}{2}}F_{l-1}^\lambda(\theta, t) \end{aligned} \tag{3.2}$$

where

$$R=e^t\left[-\cos\theta\frac{\partial}{\partial t}-\sin\theta\frac{\partial}{\partial\theta}\right],\qquad L=e^{-t}\left[-\cos\theta\frac{\partial}{\partial t}+\sin\theta\frac{\partial}{\partial\theta}\right] \tag{3.3}$$

and

$$\begin{aligned} F_l^\lambda(\theta, t)&=e^{(l+\lambda)t}(\sin\theta)^\lambda G_l^\lambda(\cos\theta),\\ G_l^\lambda(\cos\theta)&=\left[\frac{\Gamma(2\lambda)\,l!}{\Gamma(l+2\lambda)}\right]^{\frac{1}{2}}P_l^\lambda(\cos\theta). \end{aligned} \tag{3.4}$$

The Lie Algebra generated by R and L is given by the commutation laws:

$$[R, L]\equiv M=-2\frac{\partial}{\partial t};\quad [M, R]=-2R;\quad [M, L]=2L.$$

Now the existence of a particular pair of generating relations for the Ultraspherical polynomials is possible due to the relation $[R, L]\neq 0$. Thus applying the operator $(\exp vL)(\exp uR)$ to $F_l^\lambda(\theta, t)$, we obtain

$$\begin{aligned} &\sum_{n=0}^{+\infty}\binom{l+n}{n}(-ue^t)^n\sum_{m=0}^{l+n}\frac{\Gamma(l+n+2\lambda)}{\Gamma(l+n+2\lambda-m)}\frac{(-ve^{-t})^m}{m!}P_{l+n-m}^\lambda(\cos\theta)\\ &=N^l(\overline{M})^{-l-2\lambda}P_l^\lambda\left(\frac{\cos\theta-v\,e^{-t}+ue^tN^2}{N\overline{M}}\right), \end{aligned} \tag{3.5}$$

where $N^2=1-ve^{-t}\cos\theta+v^2e^{-2t}$ and $\overline{M}^2=(1-uv)^2+2u(1-uv)e^t\cos\theta+u^2e^{2t}$.

Now putting $-ue^t=x$ and $-ve^{-t}=y$, we derive

$$\begin{aligned} &\sum_{n=0}^{+\infty}\binom{l+n}{n}x^n\sum_{m=0}^{l+n}\frac{\Gamma(l+n+2\lambda)}{\Gamma(l+n+2\lambda-m)}\frac{y^m}{m!}P_{l+n-m}^\lambda(\cos\theta)\\ &=\rho_1^{\,l/2}(\omega_1)^{-\frac{l}{2}-\lambda}P_l^\lambda\left(\frac{\cos\theta+y-x\rho_1}{(\rho_1\,\omega_1)^{1/2}}\right), \end{aligned} \tag{3.6}$$

where $\rho_1=1+2y\cos\theta+y^2$ and $\omega_1=(1-xy)^2-2x(1-xy)\cos\theta+x^2$.

On the other hand, if we apply the operator $(\exp uR)(\exp vL)$ to $F_l^\lambda(\theta, t)$, we may similarly obtain

$$(3.7)\qquad \sum_{m=0}^{+\infty} x^m \sum_{n=0}^{l+m} \binom{l+m-n}{m} \frac{\Gamma(l+2\lambda)}{\Gamma(l+2\lambda-n)} \frac{y^n}{n!} P_{l+m-n}^\lambda(\cos\theta)$$

$$= (\omega_2)^{l/2} (\rho_2)^{-\frac{l}{2}-\lambda} P_l^\lambda\left(\frac{\cos\theta - x + y\rho_2}{(\rho_2\omega_2)^{1/2}}\right),$$

where $\rho_2 = 1 - 2x\cos\theta + x^2$ and $\omega_2 = (1-xy)^2 + 2y(1-xy)\cos\theta + y^2$.

Thus the formulas (3.6) and (3.7) are the desired pair of generating relations for Ultraspherical polynomials. These formulas may be considered as novel extensions of the well-known generating relation (1.3). Indeed, when $y=0$, both reduce to (1.3). Furthermore, when $x=0$, both yield the following summation formula

$$(3.8)\qquad \sum_{m=0}^{l} \frac{\Gamma(l+2\lambda)}{\Gamma(l+2\lambda-m)} \frac{y^m}{m!} P_{l-m}^\lambda(\cos\theta)$$

$$= (1 + 2y\cos\theta + y^2)^{l/2} P_l^\lambda\left(\frac{\cos\theta + y}{(1+2y\cos\theta+y^2)^{1/2}}\right).$$

Now it is clear that by LIE Algebra methods, not only the known properties of Ultraspherical polynomials can be derived, but also some of its new properties may be found out.

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