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386. THE EVALUATION OF CHARACTER SERIES BY CONTOUR INTEGRATION*

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A classical and well known application of the calculus of residues occurs in the evaluation of series of the form,

$$\sum_{n=-\infty}^{+\infty} f(n) \quad \text{or} \quad \sum_{n=-\infty}^{+\infty} (-1)^n f(n),$$

where f is a suitable meromorphic function. See the texts by HILLE [**4**, pp. 258—264] and MITRINOVIĆ [**6**, pp. 80—87] for good discussions of this topic. In this paper we extend this theory by showing how to evaluate by contour integration character series of the form,

$$\sum_{n=-\infty}^{+\infty} \chi(n) f(n) \quad \text{or} \quad \sum_{n=-\infty}^{+\infty} (-1)^n \chi(n) f(n),$$

where χ is a primitive character modulo k.

Let $G(z, \chi)$ denote the GAUSSian sum,

$$G(z, \chi)=\sum_{j=1}^{k-1} \chi(j)\, e^{2\pi i z j/k},$$

and put $G(\chi)=G(1, \chi)$. For primitive characters, we have the factorization theorem [**1**, p. 312],

$$G(n, \overline{\chi})=\chi(n)\, G(\overline{\chi}), \tag{1}$$

where n is an integer. We also put

$$\delta=\frac{1}{2}\{1-\chi(-1)\}.$$

Finally, $R\{g(z), z_0\}$ denotes the residue of $g(z)$ at $z=z_0$.

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Theorem 1. *Let f be meromorphic in the extended complex plane. Suppose that there exist positive numbers A and $a>1$ such that $|f(z)|\leqq A|z|^{-a}$, uniformly as $|z|$ tends to ∞. Let $S=S(f)=\{z_1, \ldots, z_m\}$ denote the set of all poles of f. Then,*

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{+\infty} \chi(n) f(n) = -\sum_{r=1}^{m} R\{\pi e^{-\pi i z} f(z)\, G(z, \bar{\chi})/G(\bar{\chi}) \sin(\pi z), z_r\}. \tag{2}$$

Theorem 2. *Let f and S be as given in Theorem 1. Define*

$$F(z, \chi) = \sum_{j=1}^{[k/2]} \chi(j)\, e^{2\pi i z j/k} + (-1)^{\delta} \sum_{j=1}^{[(k-1)/2]} \chi(j)\, e^{-2\pi i z j/k}, \tag{3}$$

where $[x]$ denotes the greatest integer $\leqq x$. Then,

$$\sum_{\substack{n=-\infty \\ n\notin S}}^{+\infty} (-1)^n \chi(n) f(n) = -\sum_{r=1}^{m} R\{\pi f(z) F(z, \bar{\chi})/G(\bar{\chi}) \sin(\pi z), z_r\}. \tag{4}$$

***Proof of Theorem* 1.** Let C_N denote the square whose center is the origin and whose sides are parallel to the real and imaginary axes and are of length $2N+1$, where N is an integer chosen large enough so that S is contained on the interior of C_N. By the residue theorem,

$$\begin{aligned}\frac{1}{2\pi i} \int_{C_N} \frac{\pi e^{-\pi i z} f(z)\, G(z, \bar{\chi})}{G(\bar{\chi}) \sin(\pi z)}\, dz &= \sum_{r=1}^{m} R\{\pi e^{-\pi i z} f(z)\, G(z, \bar{\chi})/G(\bar{\chi}) \sin(\pi z), z_r\} \\ &\quad + \sum_{\substack{n=-N \\ n\notin S}}^{N} R\{\pi e^{-\pi i z} f(z)\, G(z, \bar{\chi})/G(\bar{\chi}) \sin(\pi z), n\}.\end{aligned} \tag{5}$$

If $n\notin S$, we have from (1)

$$R\{\pi e^{-\pi i z} f(z)\, G(z, \bar{\chi})/G(\bar{\chi}) \sin(\pi z), n\} = f(n)\, G(n, \bar{\chi})/G(\bar{\chi}) = f(n)\, \chi(n). \tag{6}$$

By our choice of C_N, we see that there exists a positive constant $M=M(\chi)$ such that for $z=x+iy$ on C_N,

$$\left| \frac{e^{-\pi i z} G(z, \bar{\chi})}{\sin(\pi z)} \right|^2 = \frac{e^{2\pi y} \sum\limits_{j_1, j_2=1}^{k-1} \bar{\chi}(j_1)\, \chi(j_2)\, e^{2\pi i x (j_1-j_2)/k - 2\pi y (j_1+j_2)/k}}{\sin^2(\pi x) + \sinh^2(\pi y)} \leqq M.$$

Thus, the modulus of the integral on the left side of (5) is less than

$$\frac{4(2N+1)\pi A M}{|G(\bar{\chi})| \left(N+\frac{1}{2}\right)^a}.$$

Hence, upon substituting (6) into (5) and then letting N to ∞, we obtain (2).

***Proof of Theorem* 2.** Proceed as in the proof of Theorem 1. In this case we have for $n \notin S$,

(7) $$R\{\pi f(z) F(z, \bar{\chi})/G(\bar{\chi}) \sin(\pi z), n\} = (-1)^n f(n) F(n, \bar{\chi})/G(\bar{\chi}).$$

If we replace j by $k-j$ in the second sum on the right side of (3), we have by the periodicity of χ,

(8) $$F(n, \bar{\chi}) = \sum_{j=1}^{[k/2]} \bar{\chi}(j) e^{2\pi i n j/k} + (-1)^\delta \sum_{j=1+[k/2]}^{k-1} \bar{\chi}(-j) e^{2\pi i n j/k} = G(n, \bar{\chi}),$$

since $\chi(-j) = \chi(-1)\chi(j)$. By an argument similar to that in the proof of Theorem 1, there exists a constant $M = M(\chi)$ such that for z on C_N,

$$\left| \frac{F(z, \bar{\chi})}{\sin(\pi z)} \right| \leqq M.$$

Proceeding as in the previous proof, and using (8) in (7), we obtain (4) upon letting N tend to ∞.

The hypotheses on f in the above theorems may be relaxed somewhat. We could prove a similar theorem if f were meromorphic only in the finite complex plane. The growth conditions on f may also be weakened in some cases. For example, see [**4**, pp. 260—263].

EXAMPLE 1. Let $f(z) = 1/z^2$ in Theorem 1. If m is a positive integer, define

$$M_m(\chi) = \sum_{j=1}^{k-1} \chi(j) j^m.$$

Observe that for this example the left side of (2) is 0 if χ is odd. Therefore, assume that χ is even. Replacing j by $k-j$, we have since χ is even,

$$M_1(\chi) = \sum_{j=1}^{k-1} \chi(-j)(k-j) = -\sum_{j=1}^{k-1} \chi(-j) j = -M_1(\chi).$$

Thus, $M_1(\chi) = 0$. A simple calculation shows that

$$R\{\pi e^{-\pi i z} G(z, \bar{\chi})/z^2 G(\bar{\chi}) \sin(\pi z), 0\} = -2\pi^2 M_2(\bar{\chi})/k^2 G(\bar{\chi}).$$

Now [**1**, p. 313],

(9) $$|G(\chi)|^2 = k.$$

Since χ is even, from (9) we find that $G(\bar{\chi}) = k/G(\chi)$. Hence, by Theorem 1 we conclude that

$$L(2, \chi) = \sum_{n=1}^{+\infty} \chi(n) n^{-2} = \pi^2 G(\chi) M_2(\bar{\chi})/k^3.$$

If we let $f(z) = 1/z^2$ in Theorem 2, an almost identical calculation gives for χ even,

$$\sum_{n=1}^{+\infty} (-1)^n \chi(n) n^{-2} = \frac{\pi^2}{6k^3} G(\chi) \{12 N_2(\bar{\chi}) - k^2 N_0(\bar{\chi})\},$$

where

$$N_m(\chi) = \sum_{j=1}^{[k/2]} \chi(j) j^m.$$

Example 2. Let $f(z)=1/z^3$. If χ is even, the left sides of (2) and (4) are both zero in this case. Thus, assume that χ is odd. By replacing j by $k-j$, it is easy to show that for χ odd, $M_2(\chi)=kM_1(\chi)$. Using this fact, by an elementary division of power series, we find that

$$R\{\pi e^{-\pi iz} G(z,\overline{\chi})/z^3 G(\overline{\chi}) \sin(\pi z), 0\} = \frac{4\pi^3 i}{3k^3 G(\overline{\chi})} \{k^2 M_1(\overline{\chi}) - M_3(\overline{\chi})\}.$$

Since χ is odd, from (9) we find that $G(\chi) = -k/G(\overline{\chi})$. Hence, by Theorem 1 we have shown that

$$L(3,\chi) = \sum_{n=1}^{+\infty} \chi(n) n^{-3} = \frac{2\pi^3 i}{3k^4} G(\chi) \{k^2 M_1(\overline{\chi}) - M_3(\overline{\chi})\}.$$

In a similar fashion we find that

$$R\{\pi F(z,\overline{\chi})/z^3 G(\overline{\chi}) \sin(\pi z), 0\} = \frac{2\pi^3 i}{3k^3 G(\overline{\chi})} \{k^2 N_1(\overline{\chi}) - 4N_3(\overline{\chi})\}.$$

Thus, by Theorem 2,

$$\sum_{n=1}^{+\infty} (-1)^n \chi(n) n^{-3} = \frac{\pi^3 i}{3k^4} G(\chi) \{k^2 N_1(\overline{\chi}) - 4N_3(\overline{\chi})\}.$$

It is clear from the above examples that Theorem 1 enables us to calculate $L(n,\chi)$, $n \geqq 2$, when $n \equiv \delta \pmod 2$. (In fact, a slight extension of Theorem 1 enables us to calculate $L(1,\chi)$ as well, if χ is odd.) For other general methods of calculating $L(n,\chi)$ when $n \equiv \delta \pmod 2$, see **[2]**, **[3]** and **[5]**.

Example 3. Let $f(z)=1/(z^2+a^2)$, $a>0$. If χ is odd, the left sides of (2) and (4) are clearly equal to 0. Thus, assume that χ is even. A simple calculation yields

$$R\{\pi e^{-\pi iz} G(z,\overline{\chi})/(z^2+a^2) G(\overline{\chi}) \sin(\pi z), \pm ai\} = -\frac{\pi e^{\pm\pi a} G(\pm ai, \overline{\chi})}{2aG(\overline{\chi}) \sinh(\pi a)}.$$

After simplification, we find that Theorem 1 gives

$$\sum_{n=1}^{+\infty} \frac{\chi(n)}{n^2+a^2} = \frac{\pi}{2aG(\overline{\chi}) \sinh(\pi a)} \sum_{j=1}^{k-1} \overline{\chi}(j) \cosh(\pi a - 2\pi aj/k).$$

Secondly,

$$R\{\pi F(z,\overline{\chi})/(z^2+a^2) G(\overline{\chi}) \sin(\pi z), \pm ai\} = -\frac{\pi F(\pm ai, \overline{\chi})}{2aG(\overline{\chi}) \sinh(\pi a)}.$$

Thus, Theorem 2 yields upon a little simplification,

$$\sum_{n=1}^{+\infty} \frac{(-1)^n \chi(n)}{n^2+a^2} = \frac{\pi}{aG(\overline{\chi}) \sinh(\pi a)} \sum_{j=1}^{[k/2]} \overline{\chi}(j) \cosh(2\pi ja/k).$$

Observe that by letting a tend to 0 in the two results of Example 3, we obtain, respectively, the two results of Example 1.

Example 4. Let $f(z)=1/(z+a)^2$, where a is not an integer. Upon calculating the residue at $z=-a$ and using Theorem 1, we find that

$$(10) \qquad \sum_{n=-\infty}^{+\infty} \chi(n)(n+a)^{-2} = \frac{\pi^2 e^{\pi ia}}{G(\overline{\chi}) \sin(\pi a)} \sum_{j=1}^{k-1} \overline{\chi}(j) e^{-2\pi iaj/k} \{\cot(\pi a) - i + 2ji/k\}.$$

As a particular example, let $\chi(n)$ be the residue class chracter $(-4|n)$. Then, $\chi(1)=1$, $\chi(3)=-1$, $\chi(2)=\chi(4)=0$, and $G(\chi)=2i$. Replacing n by $2n+1$, we find that (10) yields after some simplification

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n}{(2n+1+a)^2} = \frac{\pi^2}{4\sin(\pi a/2)} \left\{1 - \frac{1}{\cos^2(\pi a/2)}\right\}.$$

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