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385. ON WEIGHTED MEANS OF ORDER r*

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In 1955, L. C. Hsu [4] proposed as a problem the proof of the following inequality

$$(1)\qquad 1<\frac{M(t)-M(r)}{M(t)-M(s)}<\frac{s(t-r)}{r(t-s)}\qquad (0<r<s<t),$$

involving the (ordinary) mean $M(r)$ of order r (MITRINOVIĆ [6, 75]). Apparently no proof of the inequality (1) has yet been published. The author would like to thank Professor MITRINOVIĆ for drawing his attention to this problem recently. The proof of (1) which follows is based on a little-known property of M, namely that *the function* $M(r^{-1})$ *is convex for* $r>0$. This property of M is not noted in any of the three standard references [2], [3], [6] on inequalities. However, according to E. F. BECKENBACH [1, 449], it is shown in G. JULIA [5] that $\log M(r)$ is a convex function of $1/r$ for $r>0$. Since the convexity of the logarithm of a function obviously implies the convexity of the function (because e^x is an increasing, convex function), the convexity of $M(r^{-1})$ for $r>0$ follows. Because we believe this property should be more widely known (and its proof more readily accessible), we shall include a proof of this result here.

The inequality (1) is valid not only for the (ordinary) mean of order r, but also for the weighted means of order $r>0$,

$$(2)\qquad M(r)=M_n^{[r]}(a;\ q)\equiv\left\{\sum_{k=1}^{n} q_k a_k^{\,r}\right\}^{1/r},$$

$$(3)\qquad M(r)=M^{[r]}(f;\ q;\ a,\ b)\equiv\left\{\int_a^b q(x)f(x)^r\,dx\right\}^{1/r}.$$

In (2), $a=(a_1, \ldots, a_n)$ is any sequence of positive numbers, while $q=(q_1, \ldots, q_n)$ is a sequence of positive numbers such that $\sum_{k=1}^{n} q_k=1$. Similarly in (3), f and q are posilive continuous functions on $[a, b]$ with

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$\int_a^b q(x)\,dx = 1$. The ordinary means of order r are the special cases of (2) obtained by setting all $q_k = \frac{1}{n}$. We note that if a is a constant sequence (f is a constant function) or if $n=1$, then $M(r)$ is a constant, and the middle term in (1) is not defined. For this reason we shall always assume that a is not a constant sequence (that f is not constant), and that $n \geqq 2$. In this case it is well-known (see [**2**, 16—18], [**3**, 26—27] and [**6**, 74—76]) that $M(r)$ is a continuous, strictly increasing function of r, and that $r \log M(r)$ is a strictly convex function of r. The left-hand inequality in (1) follows at once from the first of these results, and we shall now show that the strict convexity of $M(r^{-1})$ for $r>0$ follows from the second. In what follows, we deal with (2), the proof for (3) being essentially identical.

In order to prove that $f(r) \equiv M(r^{-1})$ is strictly convex for $r>0$, it suffices [**3**, 77] to show that $f''(r)>0$ for $r>0$, or since

$$f'(r) = -r^{-2} M'(r^{-1}), \qquad f''(r) = r^{-4} M''(r^{-1}) + 2\,r^{-3} M'(r^{-1}),$$

it suffices to prove that

$$g(y) \equiv y^2 M''(y) + 2\,yM'(y) > 0 \quad \text{for} \quad y>0. \tag{4}$$

Now we have

$$M'(y) = y^{-1}\left\{M^{1-y}(y) \sum_{k=1}^{n} q_k a_k^y \log a_k - M(y) \log M(y)\right\}, \tag{5}$$

and so

$$M''(y) = -y^{-2}\left\{M^{1-y} \sum_1^n q_k a_k^y \log a_k - M \log M\right\}$$
$$+ y^{-1}\left\{(1-y) M^{-y} M' \sum_1^n q_k a_k^y \log a_k + M^{1-y} \sum_1^n q_k a_k^y (\log a_k)^2 - M' - M' \log M\right\}. \tag{6}$$

From (4)—(6) it follows that

$$g(y) = M\left\{(\log M)^2 - 2\,M^{-y} \log M \sum_1^n q_k a_k^y \log a_k + M^{-2y}\left(\sum_1^n q_k a_k^y \log a_k\right)^2\right\}$$
$$+ y\left\{M^{1-y} \log M \sum_1^n q_k a_k^y \log a_k - M^{1-2y}\left(\sum_1^n q_k a_k^y \log a_k\right)^2\right.$$
$$\left. + M^{1-y} \sum_1^n q_k a_k^y (\log a_k)^2\right\}. \tag{7}$$

The first term of (7) is

$$M\left(\log M - M^{-y} \sum_1^n q_k a_k^y \log a_k\right)^2,$$

and so it suffices to prove that the second term of (7) is positive. To prove this we use the strict convexity of $h(y)\equiv y\log M(y)$ (see the suggested proof of this in [2, p. 18]), or rather the fact that

$$h''(y)=2\frac{M'}{M}+y\frac{MM''-M'^2}{M^2}>0,$$

$$2MM'+yMM''-yM'^2>0 \quad \text{for all } y. \tag{8}$$

Substituting from (5) and (6) into (8) we obtain finally

$$M^{2-y}\log M\sum_1^n q_k a_k^y\log a_k-M^{2-2y}\left(\sum_1^n q_k a_k^y\log a_k\right)+M^{2-y}\sum_1^n q_k a_k^y(\log a_k)^2>0$$

for all y. It follows from this that the second term of (7) is positive for all $y>0$, completing the proof of (1).

We note in passing that, since $(yM(y))''=yM''+2M'=y^{-1}(y^2M''+2yM')$, *it follows that $yM(y)$ is also strictly convex for $y>0$.* (This also follows from 3, TH. 119]).

The proof of the right hand inequality of (1) is now almost immediate since it is equivalent to

$$M(s)<\frac{r(t-s)}{s(t-r)}M(r)+\frac{t(s-r)}{s(t-r)}M(t), \qquad 0<r<s<t. \tag{9}$$

Setting $\lambda=t(s-r)/\{s(t-r)\}$, we have $0<\lambda<1$ for the s in question. Hence, setting $r=1/y$, $t=1/x$, $s=1/\{\lambda x+(1-\lambda)y\}$, we see that (9) is equivalent to

$$M\left(\frac{1}{\lambda x+(1-\lambda)y}\right)<\lambda M\left(\frac{1}{x}\right)+(1-\lambda)M\left(\frac{1}{y}\right), \quad (0<x<y, \quad 0<\lambda<1), \tag{10}$$

and this follows from the strict convexity of $M(r^{-1})$ for $r>0$.

We also note that, using the strict convexity of $yM(y)$ for $y>0$, one can prove that M satisfies the inequality

$$(t-r)M(rt/s)<(s-r)M(r)+(t-s)M(t) \qquad (0<r<s<t),$$

or, setting $r=\alpha s$, the inequality

$$M(\alpha t)<\frac{(1-\alpha)s}{t-\alpha s}M(\alpha s)+\frac{t-s}{t-\alpha s}M(t) \qquad (0<s<t, \quad 0<\alpha<1). \tag{11}$$

L. C. Hsu also asserted that the inequalities (1) were *best possible.* Using the continuity of M at r this is obvious for the left hand inequality of (1) if we let s approach r. We shall prove that both inequalities of (1) are best possible for arbitrary (fixed) r, s, t such that $0<r<s<t$. To prove this we first take $a_1=a_2=\cdots=a_{n-1}=\varepsilon>0$, $a_n=a>0$, and note that for each $n\geqq 2$,

$$M(r)=\left\{\varepsilon^r\sum_1^{n-1}q_k+q_na^r\right\}^{1/r}\to q_n^{1/r}a \quad \text{as } \varepsilon\to 0+.$$

It follows that

$$\frac{M(t)-M(r)}{M(t)-M(s)}\to\frac{q_n^{1/t}-q_n^{1/r}}{q_n^{1/t}-q_n^{1/s}}=\frac{1-y^{R-T}}{1-y^{S-T}}=A(y) \quad \text{as } \varepsilon\to 0+, \tag{12}$$

where $R=r^{-1}$, $S=s^{-1}$, $T=t^{-1}$ (so $R>S>T>0$), and $y=q_n$. Now, $\lim_{y\to 0+} A(y)=1$, which shows that the constant 1 appearing in the first part of (1) can not be increased, provided $q_n \to 0$ as $n\to +\infty$. However at least one of the weights q_k must tend to zero as $n\to +\infty$, and we clearly may assume this is q_n. For the second part of (1), we take $a_1=a_2=\cdots=a_{n-1}=a>0$, and $a_n=\varepsilon>0$, so that $M(r)\to\left(\sum_1^{n-1} q_k\right)^{1/r} a$ as $\varepsilon\to 0+$, and again (12) holds, but now with $y=\sum_1^{n-1} q_k=1-q_n\to 1$ as $n\to+\infty$. Now, however,

$$\lim_{y\to 1} A(y)=\lim_{y\to 1}\frac{-(R-T)\,y^{R-T-1}}{-(S-T)\,y^{S-T-1}}=\frac{R-T}{S-T}=\frac{s(t-r)}{r(t-s)},$$

and it follows that the constant on the right side of (1) can not be decreased.

At the same time that Hsu proposed the problem (1), he also proposed the inequality

$$1<Q(r)\equiv\frac{M(r)-M(-r)}{M(r)-M(0)}<n \qquad (0<r<+\infty), \tag{13}$$

where $M(r)$ is the ordinary mean defined for all $r\neq 0$ by (2) with all $q_k=1/n$, and $M(0)$ is the gemetric mean $(a_1\cdots a_n)^{1/n}$. Again, if $n\geqq 2$ and not all a_k are equal, the left hand inequality in (13) is trivially true because M is a continuous, strictly increasing function of r for $-\infty\leqq r\leqq+\infty$. Although we have not succeeded in proving the right inequality in (13), we can show that if the inequality is valid, then the constant n is best possible. Indeed, if we arrange the a_k so that $0<a_1\leqq\cdots\leqq a_n$, with strict inequality holding in at least one place, then [6, 74]

$$\lim_{r\to+\infty}\frac{M(r)-M(-r)}{M(r)-M(0)}=\frac{a_n-a_1}{a_n-(a_1\cdots a_n)^{1/n}}. \tag{14}$$

If we set $a_1=\alpha^n a_n\,(0<\alpha<1)$, and $a_2=\cdots=a_n$, the right side of (14) reduces to $(1-\alpha^n)/\,1-\alpha)$, which tends to n as $\alpha\to 1-$, so that the constant n in (13) can not be replaced by any smaller number. We observe that, using the inequality of the arithmetic and geometric means, it is easy to prove that the right side of (14) is strictly less than n. Hence if $Q(r)$ were a nondecreasing function the inequality (13) would follow! Unfortunately this is not the case since (for example) with $n=2$, $a_1=1$, $a_2=3$, one finds that $Q(1)<Q(1/2)$.

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