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# 384. AN INTEGRAL INEQUALITY FOR CONVEX FUNCTIONS*

*Alexandru Lupaş*[1]

In this paper we denote by $F$ one of the following functionals which are well-defined by the relations

$$F(f):=\frac{1}{b-a}\int_a^b f(x)\,dx,\quad F(f):=\frac{\int_a^b p(x)f(x)\,dx}{\int_a^b p(x)\,dx}\qquad (a<b),$$

(1)
$$F(f):=\sum_{k=0}^{n} p_k f(w_k)\quad (w_k\in[a,b];\ k=0,1,\ldots,n),$$

where $p:[a,b]\to\mathbf{R}$ is a positive, integrable function on $[a,b]$. Likewise we suppose that $p_k\geqq 0$ $(k=0,1,\ldots,n)$, $\sum_{k=0}^{n} p_k=1$. Clearly, if $F$ is in such a manner defined then $F(1)=1$. Sometimes instead of $F$ we write $F_x$ in order to put in evidence the corresponding variable. For instance

$$F_x(f)=\frac{1}{b-a}\int_a^b f(x)\,dx\quad\text{and}\quad F_x\big(f(z)\big)=f(z).$$

**Lemma.** *If $f, g:[a,b]\to\mathbf{R}$ are convex functions on the interval $[a,b]$, then*

(2)
$$\begin{aligned}&F(fg)\,[F(e^2)-F(e)^2]-F(f)\,F(g)\,F(e^2)\\ &\geqq F(ef)\,F(eg)-[F(f)\,F(eg)+F(g)\,F(ef)]\cdot F(e)\end{aligned}$$

*where $e(x)=x$, $x\in[a,b]$. If $f$ or $g$ is a linear function then the equality holds in* (2).

***Proof.*** Let $[x,y,z;f]$ be the divided difference of a certain function $f$. Under our conditions, for all distinct points $x,y,z$ from $[a,b]$

$$[x,y,z;f]\cdot[x,y,z;g]\geqq 0$$

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which is equivalent with

$$(3)\qquad [(y-z)f(x)-(x-z)f(y)+(x-y)f(z)]\times[(y-z)g(x)-(x-z)g(y)+(x-y)g(z)]\geqq 0$$

with equality if one of the functions is linear.

We can now make use of the fact that $F$ is a linear postitive functional; by applying successively on (3) the functionals $F_x$, $F_y$ and then $F_z$, we obtain the inequality (2). For instance if

$$A=A(x,y,z,f,g):=(y-z)(x-y)f(x)g(z)$$

then

$$F_x(A)=[F(ef)-yF(f)]\cdot(y-z)g(z),$$

$$F_zF_yF_x(A)=F(e)[F(f)F(eg)+F(g)F(ef)]-F(ef)F(eg)-F(e^2)F(f)F(g).$$

For the case in which $F$ is defined by (1), in (3) we can take $x=w_k$, $y=w_j$, $z=w_s$ $(k,j,s=0,1,\ldots,n)$.

**Theorem.** *If $f$, $g$ are convex functions on the interval $[a,b]$, then*

$$(4)\qquad \int_a^b f(x)g(x)\,dx-\frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$$

$$\geqq\frac{12}{(b-a)^3}\left(\int_a^b\left(x-\frac{a+b}{2}\right)f(x)\,dx\right)\left(\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx\right)$$

*with equality when at least one of the functions $f,g$ is a linear function on $[a,b]$.*

***Proof.*** We select

$$F(f)=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Then

$$F(e)=\frac{a+b}{2},\quad F(e^2)=\frac{a^2+ab+b^2}{3}$$

and from (2) we conclude with (4). It is clear that (2) may be written for other forms of the functional $F$, which were previously mentioned.

**Corollary.** *Let $f,g$ are convex functions on $[a,b]$ and assume that*

$$g\left(\frac{a+b}{2}-x\right)=g\left(\frac{a+b}{2}+x\right),\quad x\in\left[-\frac{b-a}{2},\frac{b-a}{2}\right].$$

*Then holds Chebyshev's inequality*

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)\leqq(b-a)\int_a^b f(x)g(x)\,dx.$$

***Proof.*** In (4) we use the fact that

$$\int_a^b \left(x-\frac{a+b}{2}\right) g(x)\,dx=0.$$

See [**1**, pp. 38-42] where there are other conditions listed under which CHEBYSHEV'S inequality is valid.

## REFERENCE

1. D. S. MITRINOVIĆ (saradnik P. M. VASIĆ): *Analitičke nejednakosti.* Beograd, 1970; (translated as *Analytic Inequalities.* Grundlehren der mathematischen Wissenschaften, Bd. 165, Berlin—Heidelberg—New York 1970)

Institutul de calcul—Cluj al Academiei R. S. România
and Mathematisches Institut A der Universitat—Stuttgart

## COMMENT OF THE EDITORIAL COMMITTEE

F. V. ATKINSON in his paper *An inequality*, These Publications **№ 357**—**№ 380** (1971), 5—6 proved the following result:

Let $f$ and $g$ be integrable functions on $[a, b]$ such that $f''>0$, $g''>0$ on $[a, b]$, and

$$\int_a^b \left(x-\frac{1}{2}(a+b)\right) g(x)\,dx=0.$$

Then CHEBYŠEV'S inequality holds:

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) \leqq (b-a) \int_a^b f(x)\,g(x)\,dx.$$

This result follows from Theorem 1 the above paper of LUPAŞ.