PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 357 — № 380 (1971)

366. ASYMMETRIC TRIANGLE INEQUALITIES*

Murray S. Klamkin

In a previous note (Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. № 330 — № 337 (1970), 1—15), the author had derived an inequality relating the elements of two triangles. Its scope has now been extended to the master inequality $x^2+y^2+z^2 \geqq (-1)^{n+1}(2yz\cos nA+2zx\cos nB+2xy\cos nC)$ where x, y, z are real numbers, n an integer and A, B, C are angles of a triangle. There is equality iff $x/\sin nA = y/\sin nB = z/\sin nC$. For $n=1$, we get BARROW and JANIĆ's inequality; however an equivalent form was given much earlier by WOLSTENHOLME. For $n=2$, we get the inequality of KOOI, $(x+y+z)^2R^2 \geqq yza^2+zxb^2+xyc^2$; the inequality of OPPENHEIM, $(a^2x+b^2y+c^2z)^2 \geqq 16(yz+zx+xy)\Delta^2$; and one of the author, $[(ax+by+cz)/4\Delta]^2 \geqq xy/ab+yz/bc+zx/ca$. However, again an equivalent form was given much earlier by WOLSTENHOLME. In particular, for $x=y=z$, $3/2 \geqq (-1)^{n+1}\Sigma\cos nA \geqq -3$. For equality, the triangle is not unique for $n>2$. This corrects some previous errors for the equality case (loc. cit., pp.7—10). The master inequality is then specialized in many ways to obtain numerous well known inequalities as well as a number of them which are believed to be new.

1. Introduction

If one peruses the book *Geometric Inequalities* [1] which is the most complete reference for triangle inequalities, one finds that almost all of the ones given are symmetric in form when expressed in terms of the sides a, b, c or the angles A, B, C of a given triangle. No doubt, part of reason is that the asymmetric ones are not as easy to come by. In this paper, we will derive a number of asymmetric inequalities. However, some of these will lead to symmetric ones by specialization and we will relate some of these to some of those given in [1].

Our point of departure will be from a positive semi-definite quadratic form in three variables.

2. A positive semi-definite quadratic form

In a previous note [2, p. 7], we had derived an equality involving the elements of two triangles which includes as special cases, the ones of BARROW and JANIĆ, KOOI, OPPENHEIM, TOMESCU as well as other well known ones. We now extend the scope of that inequality to

(1) $$x^2+y^2+z^2 \geqq (-1)^{n+1}\{2yz\cos nA+2zx\cos nB+2xy\cos nC\}$$

* Presented June 11, 1971 by O. BOTTEMA and D. S. MITRINOVIĆ.

where x, y, z are arbitrary real numbers, n an integer, and A, B, C are the angles of an arbitrary triangle. There is equality, if and only if,

$$\frac{x}{\sin nA} = \frac{y}{\sin nB} = \frac{z}{\sin nC}. \tag{2}$$

Inequality (1) follows immediately by writing it in the equivalent form

$$\{x+(-1)^n(y\cos nC+z\cos nB)\}^2+\{y\sin nC-z\sin nB\}^2 \geqq 0. \tag{3}$$

In applying (1), we can assign A, B, C to be any angles corresponding to a given triangle and then determine x, y, z for equality from (2). However, it seems more useful to assign x, y, z first; giving an asymmetric inequality for the case when x, y, z are not chosen symmetrically. Here, x, y, z cannot not be completely arbitrary real numbers if we wish to have the equality case in (1). It will turn out in an argument similar to BOTTEMA [3] that $|x|$, $|y|$, $|z|$ must form a triangle. Additionally the corresponding angles A, B, C (for the equality case) are not generally unique.

From (2), we have

$$\sin nA = kx, \qquad \sin nB = ky, \qquad \sin nC = kz$$

where k is real. Then,

$$(-1)^{n+1}\sin nC = \sin n(A+B) = k\{x(1-k^2y^2)^{1/2}+y(1-k^2x^2)^{1/2}\}$$

or after squaring twice and simplifying

$$(2kxyz)^2 = (x+y+z)(x+y-z)(y+z-x)(z+x-y). \tag{4}$$

Without loss of generality, we need only consider the two cases $x, y, z \geqq 0$ or $x, -y, z \geqq 0$. First, we take care of the degenerate case x or $z=0$. Here, (1) reduces to

$$x^2+y^2 \geqq (-1)^{n+1}2xy\cos nC.$$

For equality, we must have $\cos nC = \pm 1$, $x = \pm y$ and thus $|x|, |y|, 0$ form a degenerate triangle. The corresponding triangle ABC is not unique for $n>2$. For if $n = 2n'+1$,

$$C = \frac{2r\pi}{2n'+1}, \qquad A = B = \frac{\pi}{2} - \frac{r\pi}{2n'+1} \qquad (r = 0, 1, \dots, n');$$

for $n = 2n'$.

$$C = \frac{(2r-1)\pi}{2n'}, \qquad A = B = \frac{\pi}{2} - \frac{(2r-1)\pi}{4n'} \qquad (r = 1, \dots, n').$$

If $x, y, z > 0$, it follows from (4) that since $(2kxyz)^2 \geqq 0$, x, y, z must form a triangle XYZ and that $k = \pm 1/2\varrho$ where ϱ is the radius of the circumcircle of XYZ. Whence, (2) becomes

$$\frac{\sin X}{\sin nA} = \frac{\sin Y}{\sin nB} = \frac{\sin Z}{\sin nC} = \pm 1.$$

Thus,

$$nA = r\pi \pm X, \qquad nB = s\pi \pm Y, \qquad nC = t\pi \pm Z,$$

where $r+s+t=n\mp 1$, r, s, t have the same parity and such that $A, B, C \geqq 0$. A, B, C are unique only for the cases $n=1, 2$.

If $x, -y, z>0$, then it follows similarly, that $x, -y, z$ must form a triangle $XY'Z$ and then for equality in (1),

$$nA=r\pi\pm X, \qquad nB=s\pi\pm Y', \qquad nC=t\pi\pm Z$$

where $r+s+t=n\mp 1$, r, s and t have the same parity and such that $A, B, C\geqq 0$.

It is to be noted that the above conditions for equality in (1) rectify some erroneous ones given in [**2**, pp. 7—9].

Another positive indefinite quadratic form was also given previously [**2**, p. 13], but its scope is wider than had been indicated and its derivation was not as clear and simple as it could have been. It should have read:

If x, y, z are arbitrary real numbers and a, b, c are the sides of an arbitrary triangle of area Δ, then

$$\left\{\frac{ax+by+cz}{4\Delta}\right\}^2 \geqq \frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab} \tag{5}$$

with equality, if and only if,

$$\frac{x}{a(b^2+c^2-a^2)}=\frac{y}{b(c^2+a^2-b^2)}=\frac{z}{c(a^2+b^2-c^2)}.$$

Actually (5) corresponds to the special case $n=2$ of (1). To effect the conversion, just let $ax=x'$, $by=y'$, $cz=z'$ and note that

$$4\Delta^2=a^2b^2\sin^2 C=b^2c^2\sin^2 A=c^2a^2\sin^2 B, \ \cos 2\theta=1-2\sin^2\theta.$$

Another form equivalent to (5) is

$$(x+y+z)^2R^2\geqq yza^2+zxb^2+xyc^2 \tag{6}$$

(here R is the radius of the circumcircle of triangle ABC). The latter is due to Kooi [**1**, p. 121], [**4**]. It reduces to (5) by substituting $R=abc/4\Delta$, etc. Another equivalent version is

$$(a^2x+b^2y+c^2z)^2\geqq 16(yz+zx+xy)\Delta^2. \tag{6$'$}$$

This form and the corresponding equality conditions were stated without proof by Oppenheim [**5**] who also remarked that it would be an interesting exercise to see how many triangle inequalities could be deduced from it[1]. An earlier version occurs in the problem collection of Wolstenholme[2] [**6**, p. 92, № 514], i.e.,

$$(x\sin^2 A+y\sin^2 B+z\sin^2 C)^2\geqq 4(yz+zx+xy)\sin^2 A\sin^2 B\sin^2 C \tag{6$''$}$$

with equality iff $x\tan A=y\tan B=z\tan C$ (this reduces to a previous form by letting $x\sin^2 A=x'$, etc.).

[1] This reference was pointed out to the author by A. W. Walker after this paper had been submitted for publication. Any further references or comments indicated by an asterisk is to indicate that they have also been pointed out to the author by A. W. Walker.

[2] In a subsequent historical paper, the author expects to give an account of the many known triangle inequalities which have appeared earlier in this problem collection.

BARROW's inequality and its extension by JANIĆ [1, p. 23], i.e., if x, y, z are real numbers such that $xyz>0$, then

$$x\cos A+y\cos B+z\cos C \leqq \frac{yz}{2x}+\frac{zx}{2y}+\frac{xy}{2z}$$

(if $xyz<0$, the inequality is reversed) are equivalent to the special case $n=1$ of (1). Incidentally, the equivalent inequality

$$a^2x^2+b^2y^2+c^2z^2 \geqq xy(a^2+b^2-c^2)+yz(b^2+c^2-a^2)+zx(c^2+a^2-b^2) \tag{7}$$

appeared much earlier in the form (29) also as a problem [6, p. 32, № 207], but apparently had not been fully exploited (the x, y, z here are different from above).

3. Special Cases of Inequality (1)

It is rather surprising how many known and apparently some new triangle inequalities are special cases of (1). A number of these were obtained this way by KOOI (loc. cit.) and also in [2, p. 7]. But at the time the latter was written it was not fully realized how many more inequalities could be obtained. Consequently, there will be some duplication here.

If $x=y=z\neq 0$,

$$\frac{3}{2} \geqq (-1)^{n+1}\{\cos nA+\cos nB+\cos nC\} \tag{8}$$

with equality if

$$nA=\pi(r\pm 1/3), \qquad nB=\pi(s\pm 1/3), \qquad nC=\pi(t\pm 1/3),$$

where $r+s+t=n\mp 1$, r, s, t have the same parity and such that $A, B, C>0$.

A somewhat trivial companion inequality to (8) is

$$(-1)^{n+1}\{\cos nA+\cos nB+\cos nC\} \geqq -3^{1} \tag{9}$$

with equality if

$$nA=r\pi, \qquad nB=s\pi, \qquad nC=t\pi$$

where $r+s+t=n$ and r, s, t have the same parity as n.

Since,

$$\Sigma\cos 2nA=4(-1)^n\cos nA\cos nB\cos nC-1,$$

$$\Sigma\cos(2n+1)A=4(-1)^n\sin\frac{2n+1}{2}A\sin\frac{2n+1}{2}B\sin\frac{2n+1}{2}C+1,$$

we also have

$$\frac{1}{8} \geqq (-1)^{n+1}\cos nA\cos nB\cos nC \geqq -1, \tag{10}$$

$$\frac{1}{8} \geqq (-1)^n\sin\frac{2n+1}{2}A\sin\frac{2n+1}{2}B\sin\frac{2n+1}{2}C \geqq -1^{2}. \tag{11}$$

Special cases of (8)—(11) for $n=0, 1, 2$ appear in [1, Sect. 2].

[1] For $n=1$, the "-3" can be replaced by "1".
[2] For $n=0$, the "-1" can be replaced by "0".

Letting $yz=1/a'$, $zx=1/b'$, $xy=1/c'$ in (6), we obtain

$$\frac{(a'+b'+c')^2R^2}{a'b'c'} \geqq \frac{a^2}{a'}+\frac{b^2}{b'}+\frac{c^2}{c'}. \tag{12}$$

The latter inequality in which a', b', c' are restricted to be the sides of a trianle was proposed by TOMESCU [7] as a problem. Here, however, a', b', c' can be arbitrary real numbers. But in order to have equality, $|a'|$, $|b'|$, $|c'|$ must be the sides of a triangle. This follows from (4) since $x=a'/\sqrt{a'b'c'}$, etc. In particular, if $a'=b$, $b'=c$, $c'=a$, we obtain the asymmetric but cyclic inequality

$$R^2(a+b+c)^2 \geqq a^3c+b^3a+c^3b \ \{E\} \tag{13}$$

where for convenience the symbol $\{E\}$ will denote „with equality if and only if $a=b=c$". That $\{E\}$ applies here follows from the conditions for equality, i. e.

$$\frac{\sin B}{\sin 2A}=\frac{\sin C}{\sin 2B}=\frac{\sin A}{\sin 2C}.$$

As was shown before, the only solution (aside from trivial degenerate triangles, e.g., $a=b$, $c=0$) corresponds to the solution of

$$2A=\pi-B, \qquad 2B=\pi-C, \qquad 2C=\pi-A.$$

As a companion inequality to (13), we let $a'=a^3c$, $b'=b^3a$, $c'=c^3b$ to give

$$a^3c+b^3a+c^3b \geqq 8s^2r\sqrt{2Rr}. \tag{14}$$

If a', b', c' do not form a triangle, we have strict inequality; if they do, we have equality if ABC is equilateral. Whether there are any other solutions for equality depends on the uniqueness of the system (subject of course to $A+B+C=\pi$)

$$\tan A \sin C = \tan B \sin A = \tan C \sin B.^1$$

Letting $a'=b^2$, $b'=c^2$, $c'=a^2$, (12) reduces to

$$\left\{\frac{a^2+b^2+c^2}{4\Delta}\right\}^2 \geqq \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geqq 3 \text{ (stronger than [1, 4.4]).} \tag{15}$$

If ABC is obtuse, we have strict inequality; if not, we have equality for the equilateral case. Whether there are any other solutions for equality depends on the uniqueness of the system.

$$\frac{\sin^2 B}{\sin 2A}=\frac{\sin^2 C}{\sin 2B}=\frac{\sin^2 A^1}{\sin 2C}.$$

A companion inequality to (15) is obtained by letting $a'=a^2/b^2$, $b'=b^2/c^2$, $c'=c^2/a^2$. Whence,

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geqq \left\{\frac{a^2+b^2+c^2}{R^2}\right\}^{1/2}. \tag{16}$$

This is a weaker inequality than $\Sigma a^2/b^2 \geqq 3$, since $a^2+b^2+c^2=9R^2-OH^2$.

[1] Aside from degenerate triangles ($A=B=0$), it has shown subsequently by O. BOTTEMA, L. M. KELLY and the author that these equations imply the triangle is equilateral.

We now consider some special cases of (7). For equality in (7), we must have $x=y=z$. Letting $x=1/a$, $y=1/b$, $z=1/c$, we get

(17) $$a^3+b^3+c^3+3\,abc \geqq \Sigma\, c\,(a^2+b^2) \qquad \{E\}$$

or equivalently

(18) $$a^3+b^3+c^3+5\,abc \geqq (a+b)(b+c)(c+a),$$

(18)′ $$abc \geqq (b+c-a)(c+a-b)(a+b-c)^*, \qquad [\mathbf{1}, 1.3],$$

(19) $$R \geqq 2\,r.$$

Inequality (17) is also valid for any $a, b, c \geqq 0$ since it is a special case of SCHUR's inequality, i.e.,

$$a\,(a-b)(a-c)+b\,(b-c)(b-a)+c\,(c-a)(c-b) \geqq 0.$$

A companion inequality to (17) is the one of COLLINS [**1**, p. 13]

(20) $$2\,(a+b+c)(a^2+b^2+c^2) \geqq 3\,(a^3+b^3+c^3+3\,abc) \qquad \{E\}.$$

An equivalent form for (20) is

(21) $$s^2 \geqq 16\,Rr-5\,r^2.$$

The latter was shown by BLUNDON [**1**, p. 51] to be the strongest homogeneous quadratic inequality in s, r, R (i.e., of the type $s^2 \geqq F\,(R, r)$).

Letting $x=1/a^2$, $y=1/b^2$, $z=1/c^2$, then

(22) $$a^2b^2+b^2c^2+c^2a^2 \geqq 16\,\Delta^2 \qquad \{E\}, \qquad [\mathbf{1}, \text{p. } 45].$$

Letting $x=a$, $y=b$, $z=c$,

(23) $$a^4+b^4+c^4 \geqq \Sigma\, ab\,(a^2+b^2-c^2) \qquad \{E\}.$$

The latter inequality is valid for all real a, b, c since it is also equivalent to

(23)′ $$(b+c-a)^2(b-c)^2+(c+a-b)^2(c-a)^2+(a+b-c)^2(a-b)^2 \geqq 0^*.$$

If $x=b$, $y=c$, $z=a$, then

(24) $$a^2b^2+b^2c^2+c^2a^2-abc\,(a+b+c) \geqq a^3b+b^3c+c^3a-ab^3-bc^3-ca^3 \qquad \{E\}$$

or equivalently either

(25) $$2\,(ab^3+bc^3+ca^3) \geqq 4\,s^2(s^2+8\,Rr+r^2)-3\,(s^2+4\,Rr+r^2)^2$$

or

(26) $$4\,s^2(s^2-4\,Rr+r^2)-(s^2+4\,Rr+r^2)^2 \geqq 2\,(a^3b+b^3c+c^3a).$$

Also, see (13) and (14).

If $x=1/b$, $y=1/c$, $z=1/a$, then

(27) $$abc\left\{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right\} \geqq a^3+b^3+c^3+ab^2+bc^2+ca^2-a^2b-b^2c-c^2a \qquad \{E\}.$$

* See[1] p. 35.

If $x=b^2+c^2-a^2$, $y=c^2+a^2-b^2$, $z=a^2+b^2-c^2$, then

$$(28) \qquad \Sigma a^2(b^2+c^2-a^2)^2 \geqq 3(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2) \qquad \{E\}.$$

We now relate the result of LIDSKIĬ, OVSJANNIKOV, TULAIKOV and ŠABUNIN [**1**, p. 117] to that of (7). They showed that if p, q are real numbers such that $p+q=1$, then a triangle with sides a, b, c exists, if and only if

$$pa^2+qb^2>pqc^2 \qquad \text{for all } p, q.$$

From (7), we have that if a, b, c are sides of a triangle and x, y, z are arbitrary numbers then

$$(29) \qquad a^2(x-y)(x-z)+b^2(y-z)(y-x)+c^2(z-x)(z-y) \geqq 0.$$

There is equality if and only if $x=y=z$ provided ABC is non-degenerate. For if $c=a+b$, (29) reduces to

$$\{ax+by-(a+b)z\}^2 \geqq 0.$$

To establish sufficiency of the above result (and incidentally another proof for the necessity) one can either proceed as in [**1**, p. 119] or else note that the matrix

$$M_1=\frac{1}{2}\begin{Vmatrix} 2a^2 & c^2-a^2-b^2 & b^2-c^2-a^2 \\ c^2-a^2-b^2 & 2b^2 & a^2-b^2-c^2 \\ b^2-c^2-a^2 & a^2-b^2-c^2 & 2c^2 \end{Vmatrix}$$

which is associated with the quadratic form (29) is positive semi-definite [**8**, p. 74]. Also, note that

$$\begin{vmatrix} 2a^2 & c^2-a^2-b^2 \\ c^2-a^2-b^2 & 2b^2 \end{vmatrix} = (a+b+c)(a+b-c)(b+c-a)(c+a-b) \geqq 0.$$

and $\det(M_1)=0$. Thus $|a|, |b|, |c|$ form a triangle.

An analogous result also holds for the form (5). Here the associated matrix is given by

$$M_2=\begin{Vmatrix} a^2 & 8\Delta^2/ab-ab & 8\Delta^2/ca-ca \\ 8\Delta^2/ab-ab & b^2 & 8\Delta^2/bc-bc \\ 8\Delta^2/ca-ca & 8\Delta^2/bc-bc & c^2 \end{Vmatrix}$$

where

$$16\Delta^2=(a+b+c)(a+b-c)(b+c-a)(c+a-b).$$

The necessary and sufficient condition that (5) be a positive semi-definite form in x, y, z is that the three principal miners of M_2 be $\geqq 0$, i.e.,

$$|a^2| \geqq 0, \quad a^2b^2-(8\Delta^2/ab-ab)^2 \geqq 0, \quad \det(M_2) \geqq 0.$$

Again we need only consider „necessity" since we previously established the "sufficiency". Since it can be shown that $\det(M_2)$ vanishes identically, we need only consider the 2^{nd} order minor which can be rewritten as

$$(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$
$$\times\{4a^2b^2-(a+b+c)(a+b-c)(b+c-a)(c+a-b)\} \geqq 0.$$

Since the latter is obviously valid if $|a|, |b|, |c|$ form a triangle, we have another "sufficiency" proof. If the inequality is valid, then it is easy to show that

$$|a|+|b|-|c|, \quad |b|+|c|-|a|, \quad |c|+|a|-|b| \geqq 0$$

(we need only consider the two cases $a, b, c \geqq 0$ or $a, b, -c \geqq 0$).

The natural generalization of the previous two results is that the necessary and sufficient condition for form (1) to be positive semi-definite, in which the term $\cos nA$ is replaced by $\cos n \cos^{-1}(b^2+c^2-a^2)/2bc$ and symmetrically for the other cosine terms, is that $|a|, |b|, |c|$ form a triangle. For "necessity" the 2^{nd} order minor of the associated matrix M_n then satisfies

$$1-\cos^2 n \cos^{-1}(a^2+b^2-c^2)/2ab \geqq 0.$$

Consequently,

$$1 \geqq \frac{a^2+b^2-c^2}{2ab} \geqq -1$$

and thus $|a|, |b|, |c|$ form a triangle. Then

$$\det(M_n) = 1+(-1)^{n+1} \cos nA \cos nB \cos nC - \Sigma \cos^2 nA$$

vanishes identically.

Now we consider some special cases of (5). Letting $x=a^3$, $y=b^3$, $z=c^3$, we obtain

(30) $$a^4+b^4+c^4 \geqq 4\Delta(a^2b^2+b^2c^2+c^2a^2)^{1/2} \qquad \{E\}.$$

Coupling (30) with (22) yields

(31) $$a^4+b^4+c^4 \geqq 4\Delta \qquad \{E\} \qquad [\mathbf{1}, \text{p. } 45].$$

If $x=a^2$, $y=b^2$, $z=c^2$, then

(32) $$a^3+b^3+c^3 \geqq 4\Delta(ab+bc+ca)^{1/2} \qquad \{E\},$$

or equivalently

(33) $$s^2-6Rr-2r^2 \geqq 2r(s^2+4Rr+r^2)^{1/2}.$$

Another proof of (33) follows by squaring out and using the best homogeneous quadratic inequalities for s^2 [**1**, p. 51], i.e.,

(34) $$4R^2+4Rr+3r^2 \geqq s^2 \geqq 16Rr-5r^2 \qquad \{E\}.$$

If $x=1/a$, $y=1/b$, $z=1/c$, then

(35) $$3abc \geqq 4\Delta(a^2+b^2+c^2)^{1/2} \qquad \{E\},$$

or equivalently

(36) $$9R^2 \geqq 2s^2-8Rr-2r^2.$$

Another proof of (35) follows by applying (34) to (36).

If we let $x=1$, $y=1$, $z=0$, then

(37) $$(a+b)\sqrt{ab} \geqq 4\Delta$$

with equality, if and only if, $a=b=c/\sqrt{2}$. This is a stronger inequality than the following ones of SKOPEC and ŽAROV [1, p. 122]:

$$a^2-ab+b^2\geqq 2\Delta,\quad a^2+b^2\geqq 4\Delta,\quad (a+b)^2\geqq 8\Delta.$$

The above follows from the sequence of simple inequalities:

$$4(a^2-ab+b^2)\geqq 2(a^2+b^2)\geqq (a+b)^2\geqq 2(a+b)\sqrt{ab}.$$

For our final section, we give some more special cases of (1) which are associated in form with the following inequality of PEDOE. [1, p. 92]* relating the elements of two triangles ABC and $A_1B_1C_1$:

(38) $$a_1^2(b^2+c^2-a^2)+b_1^2(c^2+a^2-b^2)+c_1^2(a^2+b^2-c^2)\geqq 16\Delta_1\Delta$$

with equality, if and only if, $A_1B_1C_1\sim ABC$. If in (7), we now let $x=1/a_1^2$, $y=1/b_1^2$, $z=1/c_1^2$, we obtain

(39) $$a_1^2b_1^2c_1^2\left\{\frac{a^2}{a_1^4}+\frac{b^2}{b_1^4}+\frac{c^2}{c_1^4}\right\}\geqq \sum a_1^2(b^2+c^2-a^2)\geqq 16\Delta_1\Delta$$

with simultaneous equality if and only if both triangles are equilateral. Now letting $a_2=a_1^2$, $b_2=b_1^2$, $c_2=c_1^2$ (here we are assuming that $A_1B_1C_1$ is acute but this puts no restrictions on $A_2B_2C_2$), we obtain

(40) $$\frac{a^2}{a_2^2}+\frac{b^2}{b_2^2}+\frac{c^2}{c_2^2}\geqq \sum\frac{(b^2+c^2-a^2)}{b_2c_2}\geqq \frac{16\Delta_1\Delta}{a_2b_2c_2}.$$

We now employ the area inequality of FINSLER and HADWIGER [1, 10.3] that

$$4\Delta_1^2\geqq\sqrt{3}\,\Delta_2.$$

Whence,

(41) $$\frac{a^2}{a_2^2}+\frac{b^2}{b_2^2}+\frac{c^2}{c_2^2}\geqq\frac{8\Delta\sqrt{\Delta_2\sqrt{3}}}{a_2b_2c_2}\qquad \{E\}.$$

The latter also implies a number of well known inequalities. Letting $a_2=a$, etc. and then letting $a_2=b$, etc., and using (16) we obtain the following chain:

(42) $$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geqq 3\geqq\left\{\frac{a^2+b^2+c^2}{R^2}\right\}^{1/2}\geqq\frac{8\Delta^{3/2}\,3^{1/4}}{abc}\qquad \{E\}$$

(see [1, p. 42 (4.4)]). We obtain an alternate chain by considering (27), the l.h.s. of (42) and (40) with $a_2=a$, $b_2=b$, $c_2=c$ giving

(43) $$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\geqq 3\geqq\frac{b^2+c^2-a^2}{bc}+\frac{c^2+a^2-b^2}{ca}+\frac{a^2+b^2-c^2}{ab}\qquad \{E\},$$

and

(44) $$3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\geqq (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\qquad \{E\}.$$

The latter elegant asymmetric inequality was recently proposed as a problem by A. W. WALKER [9]. I confess I was not able to prove it until after a

* This result had been established much earlier by J. NEUBERG: *Sur les projections...* Acad. Roy. de Belgique, Mémoires Couronnés 8°, **44** (1891), 31—33.

communication with the proposer in which he casually mentioned that he had arrived at it in connection with some work on BROCARD points[1]. That triggered off the solution. For if Ω denotes a BROCARD point of ABC, then $B\Omega = (2R\sin\omega)\, b/c$, etc. [9, pp. 264—268]. Since also, $a^2b^2+b^2c^2+c^2a^2 = 4\Delta^2/\sin^2\omega$, (44) can be rewritten as

$$3(A\Omega^2+B\Omega^2+C\Omega^2) \geqq (a^2+b^2+c^2) \qquad \{E\}.$$

The latter is valid for any point Ω [1, p. 117] with equality if and only if Ω coincides with the centroid. That this only occurs for an equilateral triangle, follows from [10, p. 268, eq. d]. In a similar way, one can show (using [1, 12.18]) that

$$(45) \qquad \left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right\}^2 \geqq 4\sqrt{3}\,\Delta\left\{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right\} \qquad \{E\}.$$

We can also change the elements of (44) and (41) to their square roots without loss of generality as we did for (39). Whence,

$$(46) \qquad \frac{a}{a_2}+\frac{b}{b_2}+\frac{c}{c_2} \geqq \frac{8^{1/2}\,3^{5/8}\,\Delta^{1/2}\,\Delta_2^{1/4}}{(a_2b_2c_2)^{1/2}} \qquad \{E\},$$

$$(47) \qquad 3\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right\} \geqq (a+b+c)\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\} \qquad \{E\},$$

$$(47)' \qquad (a+b-c)(b-c)^2+(b+c-a)(c-a)^2+(c+a-b)(a-b)^2 \geqq 0^*.$$

Also, for any real a, b, c, $(abc \neq 0)$, we have

$$(48) \qquad \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geqq \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \qquad \{E\}.$$

This follows immediately from $\Sigma(a/b-b/c)^2 \geqq 0$. Also, since

$$3\left\{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right\} \geqq \left\{\left|\frac{a}{b}\right|+\left|\frac{b}{c}\right|+\left|\frac{c}{a}\right|\right\}^2 \geqq 9, \qquad (abc \neq 0),$$

$$(49) \qquad \left\{\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right\} \geqq \left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right\} \qquad \{E\},$$

where a, b, c are real and $abc \neq 0$.

We now compare some of the latter inequalities. There is no comparison between the second parts of (42) and (43). Here we are equivalently comparing $\Sigma\cos A$ and $(\Sigma\sin^2 A)^{1/2}$. Now consider the two cases, $(\pi/2, \pi/2, 0)$, $(\pi, 0, 0)$. The last part of (42) is weaker than (43) since

$$(50) \qquad a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2) \geqq 8\,\Delta^{3/2}\,3^{1/4} \qquad \{E\},$$

is equivalent to

$$(51) \qquad (R+r)^2 \geqq \sqrt{3}\,rs.$$

On using $4R+r \geqq s\sqrt{3}$ [1, p. 49], we get

$$(R+r)^2 \geqq r(4R+r) \geqq rs\sqrt{3}.$$

---

* In a further communication, the proposer notes that his proof follows by setting $(x, y, z) = (b^2, c^2, a^2)$ in (7).

Inequality (27) is stronger than the l. h. s and r. h. s of (43). This implies that

$$a^3+b^3+c^3 \geqq a^2b+b^2c+c^2a \tag{52}$$

or equivalently that

$$a^2(a-b)+b^2(b-c)+c^2(c-a) \geqq 0. \tag{53}$$

The latter is similar to a SCHUR type inequality and it can be proven in the same way. We show more generally that

$$I=a^n(a-b)+b^n(b-c)+c^n(c-a) \geqq 0 \qquad \{E\}, \tag{54}$$

for all $a, b, c \geqq 0$ and $n>0$.[1] Since I is cyclic in a, b, c, it suffices to consider the two cases $a \geqq b \geqq c$, $a \leqq b \leqq c$. In the first case,

$$I \geqq a^n(a-b)+b^n(b-c)-b^n(a-c)=(a^n-b^n)(a-b) \geqq 0.$$

In the second case,

$$I \geqq b^n(c-a)-b^n(c-b)-a^n(b-a)=(b^n-a^n)(b-a) \geqq 0.$$

In order to show that (27) is a stronger inequality than the first part of (42), we need to show that

$$a^3+b^3+c^3 \geqq 3\,abc+k\,(a-b)(b-c)(a-c) \tag{55}$$

where $k=1$. Actually, a, b, c need not be the sides of a triangle but can be any non-negative numbers. Also, $k=1$ can be replaced by $k=5/2$.[2] We may assume without loss of generality that $a \geqq b \geqq c$ and we rewrite (55) as

$$\{a+b+c\}\{(a-b)^2+(b-c)^2+(a-c^2)^2\} \geqq 5\,(a-b)(b-c)(a-c).$$

Then,

$$(a-c)^2(a+b+c)=(a-c)^2\{(b-c)+(a-b)+(2\,c+b)\} \geqq 3\,(a-b)(b-c)(a-c)$$

and

$$(a+b)^2+(b-c)^2 \geqq \left\{\frac{a-c}{a+b+c}\right\} 2\,(a-b)(b-c).$$

Although (55) does not appear to be a sophisticated extension of the A.M.-G.M. inequality for three variables, apparently it does not follow from the various extensions of the A.M.-G.M. given in [**11**, pp. 81—85].

Finally, there is no direct comparison between (27) and (44).

ACKNOWLEDGMENT. The author is very grateful to A. W. WALKER for many helpful suggestions.

---

[1] For $n<0$, the inequality is reversed.

[2] In a private communication, D. J. NEWMAN showed that the best possible constant is $\sqrt{9+6\sqrt{3}}$ .This will appear as a problem proposal in the SIAM Review, Oct. 1971.

REFERENCES

1. O. Bottema, R. Ž. Đorđević, R. R. Janić, D. S. Mitrinović, P. M. Vasić: *Geometric Inequalities*. Groningen, 1969.
2. M. S. Klamkin: *Notes on inequalities involving triangles or tetrahedrons*. These Publications № 330 — № 337 (1970), 1—15.
3. O. Bottema: *An inequality for the triangle*. Simon Stevin **33** (1959), 97—100.
4. O. Kooi: *Inequalities for the triangle*. Simon Stevin **32** (1958), 97—101.
5. *Problem E* 1724. Amer. Math. Monthly **72** (1965), 792.
6. J. Wolstenholme: *Mathematical Problems*. London, 1891.
7. I. Tomescu: *A two triangle inequality*, Amer. Math. Monthly **78** (1971), 82—83.
8. R. Bellman *Introduction To Matrix Analysis*. New York, 1960.
9. A. W. Walker: *Problem* 774, Math. Mag. **43** (1970), 226.
10. R. A. Johnson: *Advanced Euclidean Geometry*, New York, 1960.
11. D. S. Mitrinović: *Analytic Inequalities*. Berlin — Heidelberg — New York, 1970.

Scientific Research Staff
Ford Motor Company
Dearborn, Michigan