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365. BOUNDS FOR ORDER STATISTICS*

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By using a result of QUESENBERRY and DAVID [2], HAWKINS [1] has obtained bounds on the order statistics for an arbitrary statistical distribution. His results are equivalent to the following:

Theorem. *If* $\sum_{i=1}^{n} x_i = 0$ *and* $\sum_{i=1}^{n} x_i^2 = 1$ *where* $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ *then*

$$-\sqrt{\frac{n-1}{n}} \leqq x_1 \leqq -\frac{1}{\sqrt{\{n(n-1)\}}},$$

$$-\sqrt{\frac{n-j}{nj}} \leqq x_j \leqq \sqrt{\frac{j-1}{n(n+1-j)}} \quad \text{for } 2 \leqq j \leqq n-1,$$

and

$$\frac{1}{\sqrt{n(n-1)}} \leqq x_n \leqq \sqrt{\frac{n-1}{n}}.$$

all the inequalities being best possible.

Proof. The following alternative proof is independent of statistical ideas. Define θ by $\sum_{i=1}^{m} x_i = m x_m - \theta$ so that $\theta \geqq 0$.

If $1 \leqq m \leqq n-1$ and $x_m \leqq 0$ then

$$\sum_{i=1}^{n} x_i^2 \geqq \sum_{i=1}^{m} x_m^2 + \sum_{i=m+1}^{n} x_i^2$$

$$= m x_m^2 + \sum_{i=m+1}^{n} \left(x_i - \frac{\theta - m x_m}{n-m}\right)^2 + \frac{(\theta - m x_m)^2}{n-m}.$$

Since $\sum_{i=m+1}^{n} x_i = \theta - m x_m$, we have

$$\sum_{i=1}^{n} x_i^2 \geqq \frac{nm}{n-m}\left(x_m - \frac{\theta}{n}\right)^2 + \frac{\theta^2}{n} + \sum_{i=m+1}^{n} \left(x_i - \frac{\theta - m x_m}{n-m}\right)^2.$$

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Hence if $x_m < -\sqrt{\frac{n-m}{nm}}$ and $\theta \geqq 0$ then

$$\sum_{i=1}^{n} x_i^2 > \frac{mn}{n-m}\left(-\sqrt{\frac{n-m}{nm}} - 0\right)^2 + 0 + 0 = 1.$$

Hence x_m cannot be less than $-\sqrt{\frac{n-m}{nm}}$ for $m = 1$ to $n-1$.
It can, however, attain this lower bound, as is seen by taking

$$x_1 = \cdots = x_m = -\sqrt{\frac{n-m}{nm}} \text{ and } x_{m+1} = \cdots = x_n = \sqrt{\frac{m}{n(n-m)}}.$$

To show that the greatest lower bound for x_m is given by a different expression in the case $m = n$ suppose that exactly r of the x's are greater than 0. Then $r \leqq n-1$ and $x_1 \leqq x_2 \leqq \cdots \leqq x_{n-r} \leqq 0 < x_{n-r+1} \leqq x_{n-r+2} \leqq \cdots \leqq x_n$.

If $x_n < \frac{1}{\sqrt{\{n(n-1)\}}}$ then

$$0 \geqq x_1 + \cdots + x_{n-r} = -(x_{n-r+1} + \cdots + x_n) > \frac{-r}{\sqrt{n(n-1)}}$$

and

$$\sum_{i=1}^{n} x_i^2 \leqq \sum_{i=1}^{n-r} x_i^2 + 2 \sum\sum_{1 \leq i < j \leq n-r} x_i x_j + \sum_{i=n-r+1}^{n} x_i^2$$

$$\leqq \left(\sum_{i=1}^{n-r} x_i\right)^2 + \sum_{i=n-r+1}^{n} x_n^2 < \frac{r^2}{n(n-1)} + \frac{r}{n(n-1)} = \frac{r(r+1)}{n(n-1)} \leqq 1.$$

Hence x_n cannot be less than $\frac{1}{\sqrt{n(n-1)}}$; but this lower bound for x_n is attained for the set $x_1 = -\sqrt{\frac{n-1}{n}}$, $x_2 = \cdots = x_n = \frac{1}{\sqrt{n(n-1)}}$.

For upper bounds on the x's we first have, by symmetry, that the m^{th} largest of them, x_{n-m+1}, must satisfy

$$x_{n-m+1} \leqq \sqrt{\frac{n-m}{nm}} \quad \text{for } 1 \leqq m \leq n-1$$

and

$$x_{n-m+1} \leqq -\frac{1}{\sqrt{n(n-1)}} \quad \text{for } m = n,$$

and putting $m = n+1-j$ gives the stated inequalities for x_j.

REFERENCES

1. D. M. Hawkins; *On the bounds of the range of order statistics.* To appear in Jour. Amer. Stat. Assn. 1971.
2. C. P. Quesenberry and H. A. David: *Some tests for outliers.* Biometrika **48** (1961), 379—390.

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