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364. ON SOME PROPERTIES OF FUNCTION $\pi(x)$*

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The well known LANDAU hypothesis [1] asserts that the following inequality

(1) $$\pi(2x) \leqq 2\pi(x)$$

holds for every real number $x \geqq 3$.

Recently J. ROSER and L. SCHOENFELD [2] have proved the truth of the LANDAU conjecture, by demonstrating that the inequality

(1') $$\pi(2x) < 2\pi(x)$$

holds for all $x \geqq 347$.

Therefore, the LANDAU hypothesis has become a theorem for $x \geqq 347$.

Now we are going to generalize this LANDAU theorem and to prove some other theorems. With that aim we shall establish the following:

Theorem 1. *Let k be any real number bigger or equal to* $\sqrt{e}\left(\approx 1.65 = \frac{33}{20}\right)$, *i.e.*

(2) $$k \geqq \sqrt{e} \qquad (e = 2.718281828\ldots).$$

Then the following inequality (more general than (1'))

(3) $$\pi(kx) < k\,\pi(x)$$

for every $x \geqq 347$ *holds.*

Proof. Owing to J. ROSSER, L. SCHOENFELD and YOHE [3] inequalities:

$$\frac{x}{\log x - \frac{5}{4}} > \pi(x) > \frac{x}{\log x - \frac{3}{4}} \qquad (\forall\, x \geqq 347)$$

we obtain directly:

$$\frac{kx}{\log x - \frac{5}{4}} > k\,\pi(x) > \frac{kx}{\log x - \frac{3}{4}}$$

and

$$\frac{kx}{\log x + \log k - \frac{5}{4}} > \pi(kx) > \frac{kx}{\log x + \log k - \frac{3}{4}} \qquad (k \geqq \sqrt{e})$$

* Received in revised form May 10, 1971, and presented by D. S. MITRINOVIĆ.

or

$$k\pi(x) > \frac{kx}{\log x - \frac{3}{4}} \geqq \frac{kx}{\log x + \left(\log k - \frac{1}{2}\right) - \frac{3}{4}} > \pi(kx),$$

since—by condition (See (2))—we have

$$\log k - \frac{1}{2} \geqq 0.$$

So inequality (3) is proved completely for every $x \geqq 347$ and for every $k \geqq \sqrt{e}$!

Evidently, LANDAU inequality (1′) is a particular case of (3), because it can be obtained from (3) for $k = 2\,(> \sqrt{e})$.

Theorem 2. *Let k be any positive number* $\leqq \frac{1}{\sqrt{e}}$; *then inequality*

(3′) $$\pi(kx) > k\pi(x)$$

(*which is absolutely contrary of* (3)) *for all x with* $kx \geqq 347$ *holds.*

Proof. The proof is quite similar to that of theorem 1.

Theorem 3. *If x designates an arbitrary real number* $\geqq 347$ *and if* $a \geqq \sqrt[4]{e}$, *then we shall have either*

(4) $$\pi(ax) < a\pi(x)$$

or

(5) $$\pi(a^2x) < a\pi(ax).$$

Proof. If we suppose that inequality (4) does not hold, then it follows that

$$\pi(ax) \geqq a\pi(x),$$

or

(6) $$a\pi(ax) \geqq a^2\pi(x).$$

On the other hand, if we fix $k = a^2\,(\geqq \sqrt{e})$, inequality (3) will give

(7) $$a^2\pi(x) > \pi(a^2x).$$

Now owing to (6) and (7) we come to the conclusion that

$$a\pi(ax) > \pi(a^2x).$$

So theorem 3 is proved.

REFERENCES

1. D. S. MITRINOVIĆ: *Analitičke nejednakosti.* Beograd 1970.
2. J. ROSER and L. SCHOENFELD: *Abstract of brief scientific communications to the International Congress in Moscow* 1966, Section 3, p. 8.
3. J. ROSER, L. SCHOENFELD and YOHE: *Proceedings of the* 1968 *International Federation for Information Processing Congress.*