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359. AN INEQUALITY*

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The following is proved:

Let $p(x),\ q(x) \in C''[a, b]$, and let $p''>0,\ q''>0$, and also

(1)
$$\int_a^b \left(x-\frac{1}{2}(a+b)\right) q(x)\, \mathrm{d}x=0.$$

Then

(2)
$$(b-a) \int_a^b pq\, \mathrm{d}x > \int_a^b p\, \mathrm{d}x \int_a^b q\, \mathrm{d}x.$$

Let us write

(3)
$$m=(b-a)^{-1} \int_a^b q\, \mathrm{d}x,$$

so that what we have to prove may be written as

(4)
$$\int_a^b p(x)\big(q(x)-m\big)\, \mathrm{d}x>0.$$

We write further

(5)
$$q_1(x)=\int_a^x \big(q(t)-m\big)\, \mathrm{d}t, \qquad q_2(x)=\int_a^x q_1(t)\, \mathrm{d}t.$$

Integrating the left of (4) by parts twice we get

(6)
$$(pq_1-p' q_2)\Big|_a^b + \int_a^b p''\, q_2\, \mathrm{d}x.$$

Here $q_1(a)=q_2(a)=0$, and $q_1(b)=0$ since m is the mean-value of q. Thus **(6)** becomes

(7)
$$-p'(b)\, q_2(b)+\int_a^b p''\, q_2\, \mathrm{d}x.$$

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Thus to complete the proof it will be sufficient to show that

(8) $$q_2(b)=0,$$

(9) $$q_2(x)>0, \qquad a<x<b.$$

Concerning (8), we have

$$q_2(b)=\int_a^b dx\int_a^x \big(q(t)-m\big)\,dt=\int_a^b (b-t)\big(q(t)-m\big)\,dt$$

$$=\int_a^b \left(\frac{1}{2}(a+b)-t\right)\big(q(t)-m\big)\,dt+\frac{1}{2}(b-a)\int_a^b \big(q(t)-m\big)\,dt,$$

and here the first integral vanishes in view of (1), and the second by (3).

It remains to prove (9). We note first that since $q''>0$, $q(x)-m$ can have at most two zeros in $a<x<b$; by (3), it must have at least one zero in this open interval. By ROLLE's theorem, and the facts that $q_1(a)=q_1(b)=0$, we see that $q_1(x)$ can have at most one zero in $a<x<b$. Again by ROLLE's theorem, and the facts that $q_2(a)=q_2(b)=0$, we have that $q_2(x)$ can have no zero in $a<x<b$. Thus the sign of $q_2(x)$ is fixed in $a<x<b$. To establish (9) we have to show that it is positive for at least one x.

The above argument, using ROLLE's theorem, shows that $q_1=q_2'$ must have at least one zero in $a<x<b$, and so exactly one zero. Likewise, $q-m=q_1'$ has at least two zeros, and so exactly two zeros in $a<x<b$. Since $(q-m)''>0$, it must start positive, then become negative, and then positive, as x goes from a to b. Thus $q_1(x)$, $q_2(x)$, the successive integrals of $q(x)-m$, must also be positive for $x>a$ and close to a. Thus the constant sign of q_2 in $a<x<b$ is positive, and this completes the proof.

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