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358. A VOLUME INEQUALITY FOR SIMPLEXES*

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If D, E, F denote the points in which the angles bisectors of a triangle ABC meet the opposite sides, then GRIDASOV [1] has shown that the area of DEF is at most one quarter the area of ABC with equality only if ABC is equilateral. We extend this result to the following:

Theorem. *If $V_0, V_1, \ldots, V_n$ denote the $n+1$ vertices of an n-dimensional simplex S in E^n and if $V_0', V_1', \ldots, V_n'$ denote the $n+1$ vertices of an inscribed simplex S' such that the cevians V_iV_i' $(i=0, 1, \ldots, n)$ are concurrent within S, then*

$$\text{Vol. } S \geqq n^n \text{ Vol. } S'$$

with equality, if and only if, the point of concurrency of the cevians is the centroid of S.

Proof. Let $\mathbf{V}_i$ $(i=1, \ldots, n)$ denote n linearly independent vectors from V_0 to V_i and let

$$\mathbf{P}=\overrightarrow{V_0P}=\lambda_1 \mathbf{V}_1+\cdots+\lambda_n \mathbf{V}_n$$

where P denotes the point of concurrency so that

$$\lambda_i \geqq 0 \ (i=1, \ldots, n), \ \sum_{i=1}^{n} \lambda_i<1.$$

Then,

$$\mathbf{V}_i'=\mathbf{V}_i+(\mathbf{P}-\mathbf{V}_i)/(1-\lambda_i) \quad (i=1, \ldots, n),$$
$$\mathbf{V}_0'=\mathbf{P}/(1-\lambda_0)$$

where $1-\lambda_0=\lambda_1+\cdots+\lambda_n$. If the rectangular components of $\mathbf{V}_i$ are a_{ij} $(j=1, \ldots, n)$, the volume of S is given by

$$\text{Vol. } S=[\mathbf{V}_1, \ldots, \mathbf{V}_n]/n!=\frac{1}{n!}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{vmatrix}.$$

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To express Vol. S' in terms of Vol. S, we first note that a determinant is a linear function of each of its elements. Whence,

$$\text{Vol. } S' = [r_1 V_1 + s_1 P, \ldots, r_n V_n + s_n P]$$

or

$$\text{Vol. } S' = \{\text{Vol. } S\} \left\{ \prod_{i=1}^{n} r_i \right\} \left\{ 1 + \frac{s_1 \lambda_1}{r_1} + \cdots + \frac{s_n \lambda_n}{r_n} \right\}$$

where

$$r_i = \frac{\lambda_i}{1-\lambda_i}, \qquad s_i = \frac{\lambda_0 - \lambda_i}{(1-\lambda_i)(1-\lambda_0)}.$$

It is to be also noted that in setting up the determinant for the volume S', we have taken V_0' to be the new origin. Simplifying the above,

$$\text{Vol. } S' = \frac{n \lambda_0 \lambda_1 \cdots \lambda_n}{(1-\lambda_0)(1-\lambda_1) \cdots (1-\lambda_n)} \text{ Vol. } S.$$

The maximum of the λ expression subject to $\sum_{i=0}^{n} \lambda_i = 1$ is easily found by the A. M.-G. M. theorem [2]:

$$\frac{1-\lambda_i}{n} = \frac{\lambda_0 + \lambda_1 + \cdots + \lambda_n - \lambda_i}{n} \geqq \left\{ \frac{\lambda_0 \lambda_1 \cdots \lambda_n}{\lambda_i} \right\}^{1/n}.$$

Thus

$$\prod_{i=0}^{n} (1-\lambda_i) \geqq n^{n+1} \lambda_0 \lambda_1 \cdots \lambda_n$$

with equality if $\lambda_i = 1/(n+1)$. Finally,

$$\text{Vol. } S \geqq n^n \text{ Vol. } S'$$

with equality only if P is the centroid.

For the result of GRIDASOV, $n=2$. And for equality, the angle bisectors must coincide with the medians which implies that the triangle must be equilateral.

REFERENCES

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2. M. S. KLAMKIN, D. J. NEWMAN: *Extensions of the Weierstrass Product Inequalities*. Math. Mag. **43** (1970), 137—141.

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