PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 354 — № 356 (1971)

354. GRAPHS AND THEIR SPECTRA*

Dragoš M. Cvetković

INTRODUCTION

Problems involving graphs are diverse in their outher form, as well as in their essence. The graph theory has not as yet developed general methods which would enable one to solve a larger number of problems by a unique method. Many problems are still unsolved, and for some of them one does not know, even approximately, in which direction to look for the solution. Special methods are developed for different classes of problems.

In this paper we describe a set of procedures for investigations of graphs. Since it is based on the spectrum of the adjacency matrix of a graph, we suggest the name *spectral method.*

The spectral method is a general method, since it can be applied to a variety of unrelated problems. However, as we shall see later, it is not *the* general method, i.e. it cannot provide an answer to all questions that can be posed in connection with a given graph. Naturally, this method will give the best results in connection with other methods.

The spectral method belongs to a group of algebraic methods in the graph theory. The idea is to determine the structural properties of graphs by means of an algebraic invariant of graphs — the spectrum of a graph. As a starting point one uses the theorems known in matrix theory.

We shall define more precisely the notion of the spectral method.

The following concepts: vertex, edge, subgraph, connectedness, chromatic number, etc. which appeared as a result of the representation of a graph by a figure, are associated with graphs. We shall say that a proposition which involves only such concepts describes the structure of a graph.

For the purposes of investigations, graphs can be brought into connection with other mathematical objects. One such object is the spectrum of a graph. The spectrum of a graph, in a wider sense, will mean the spectrum of an arbitrary square matrix which is, in a given way, associated with a graph. Propositions

* This paper represents an abridged version of the Doctoral Dissertation, defended at the Faculty of Electrical Engineering, Beograd, on 27-th May 1971. Received on 1-st July 1971. The original version in Serbo-Croatian was completed on 15-th December 1970.

which describe the relations between the numbers contained in the spectrum of the graph and which do not involve the concepts of a structural nature, will be said to describe the spectral properties of a graph.

Theorems which describe and connect various structural properties of a graph are those which are ultimately important in the theory and applications of graphs. Theorems which provide connections between structural properties of a graph and properties of an object which is associated with the graph, are means of the graph theory, i.e. they are the methods used. We notice that a large number of results in graph theory is proved without any such method, but by a direct confrontation of structural properties whose relation is sought.

The spectral method in the graph theory will mean a set of procedures for obtaining and proving propositions involving the structure of graphs, which use essentially the spectrum of a graph.

The spectral method was founded in the last 13 years, as a result of the work of a considerable number of mathematicians. The greatest credit is due to L. Collatz, A. J. Hoffman and H. Sachs. The fundamental paper in this field was published in 1957 by L. Collatz and U. Sinogowitz **[10]**[1]. According to the bibliography we give at the end, 83 papers dealing with the spectral method were published up to now. These papers were written by 49 mathematicians and are published in 46 different journals and other publications.

We might notice that owing to this dislocation of papers, some authors were not aware of the existance of articles which are similar to those they published.

The majority of theorems in those papers are related to the connection between the structural and spectral properties of graphs. The question of investigating this connection was implicitely posed by L. Collatz and U. Sinogowitz **[10]**, and explicitely by H. Sachs **[78]**, and by A. J. Hoffman **[46]**.

While writing this paper, I have aimed at the following goals:

1° To supplement the existing procedures of the spectral method by original contributions;

2° To connect the results of various authors;

3° To show by concrete examples the possibilities of application of the spectral method;

4° To expose in one place all the important results of this discipline from a unique outlook.

The paper contains five chapters and the bibliography.

Chapter 1 is the introductory chapter. It contains the basic definitions, the problem is announced and the fundamental properties of the spectrum of a graph are described.

In Chapter 2 we start with some combinatorial problems which we then connect with the spectrum of a graph. This enables us to determine certain spectral properties of graphs.

In Chapters 3 and 4 we expose the procedures of the spectral method.

---

[1] Papers **[92]** and **[102]** appeared before this paper. **[92]** is an unpublished doctorate thesis in which the author, dealing with other things, arrives at the graph spectrum. **[102]** is a summary of a paper, which was communicated at the 3 rd Congress of Mathematicians of USSR. This paper is also not published.

Chapter 3 is devoted to the problem of identification of graphs by the use of spectra.

In Chapter 4 we give possibilities for determination of structural details of a graph by means of the spectrum. At the end of the chapter we give a list of papers which have, by the use of a spectral method, achieved certain results and we discuss the possibilities of further applications of the spectral method.

Chapter 5 describes an example of application of the spectral method. The spectral method is used to describe the graphs which are obtained as a result of operations from a class of *n*-ary operations applied to graphs.

We sometimes refer to theorems which are exposed later on in the text. The reason is that we tried to group the results into certain entities. Certain number of theorems of other authors, whose proofs can be found in the original papers, is given without proofs. The greater part of these theorems is used in our proofs, while some of them are quoted for the completeness sake.

* *
*

I use this opportunity to thank Professor D. S. Mitrinović, under whose guidance I have begun with my scientific work and who has directed this dissertation.

I am grateful to Professors L. Collatz, V. Devidé, Đ. Kurepa, S. Prešić, H. Sachs, M. Stojaković and P. Vasić who have read some or most of my papers upon which this dissertation is based, giving me very useful remarks and suggestions.

For various forms of help and cooperation I would also like to thank assistants J. Kečkić and R. Lučić.

1. PRELIMINARIES

In Section 1.1. basic definitions are given and the problem with which this paper deals is announced. Some data on the literature are supplied and basic results of other authors, in the same field, are presented.

Section 1.2. contains a description of the basic properties of a graph spectrum. Several theorems of matrix theory, which are in connection with graph spectra are listed. Basic features of the spectrum of an undirected graph without loops or multiple edges are summarized in Theorem 1.12.

1.1. The problem announced

Primarily, we shall quote definitions of some basic notions.

Definition 1.1. *Let* $X=\{x_1, \ldots, x_n\}$ *be a non-empty set and* U *a combination with repetition of the set* $X \times X$. *The ordered pair* $G=(X, U)$ *is called a graph. Elements of* X *are vertices, while elements of* U *are edges of the graph.*

A graph can obviously be represented by a figure in the following way. We represent the vertices $x_1, \ldots, x_n$ of the graph by arbitrary mutually different points in a plane or in space. Every edge $(x_i, x_j) \in U$ is represented by a continuous smooth curve joining points representing vertices x_i and x_j; this curve is oriented from x_i to x_j. The edge joining a vertex with itself is called a loop.

Definition 1.2. *The adjacency matrix* $A = \| a_{ij} \|_1^n$ *of the graph* G *is the matrix with entry* a_{ij} *equal to the number of edges leading from the vertex* x_i *to the vertex* x_j.

Graphs, whose adjacency matrix is symmetric, have the property that for arbitrary x_i, x_j, the same number of edges leads to the vertex x_j from the vertex x_i, as is the case from x_j to x_i. The set of edges of such graphs, irrespectively of loops, may be represented as a set of pairs of edges, where every pair contains the edges joining the same vertices but having the opposite orientations. Each such pair can be represented by an uniq undirected edge. Therefore, the graphs with symmetric adjacency matrix will be called undirected graphs. The undirected graphs will be represented by a figure, if necessary, in both described ways. The term "directed graph" will be applied also to the undirected graph when we mean their representation by directed graphs. A graph G is called a digraph if two edges, one leading from x_i to x_j, and the other from x_j to x_i are not existent in G for any pair of vertices x_i, x_j.

Definition 1.3. *Graphs* G_1 *and* G_2 *are isomorphic if and only if there is a one-to-one mapping* φ *of the set of vertices of* G_1 *onto the set of vertices of* G_2 *with property that for every pair of vertices* (a, b) *from* G_1 *there are so many edges of the form* (a, b) *in* G_1, *as there are edges in* G_2 *of the form* $(\varphi(a), \varphi(b))$.

Isomorphic graphs are, in fact, the identical graphs. If A_1 and A_2 are adjacency matrices of two isomorphic graphs, then there is a permutation matrix P such that $A_1 = P^{-1} A_2 P$.

Definition 1.4. *The characteristic polynomial* $P_G(\lambda) = \det(\lambda I - A)$ *of an arbitrary adjacency matrix* A *of the graph* G *is called the characteristic polynomial of the graph. The spectrum (the set of characteristic values i.e. eigenvalues) of the adjacency matrix* A *of the graph* G *is called the spectrum of the graph.*

Since all adjacency matrices are mutually similar (through permutation matrices), the characteristic polynomial of a graph is unique, i.e., it is invariant in relation to any reordering of vertices of the graph. The spectrum is also, in the described sense, an invariant of the graph.

The results of this paper are related primarily to finite, undirected graphs without loops or multiple edges. Parallelly, other kinds of graphs are dealt with, but finite graphs only are involved.

Basic questions which can be posed in connection with the spectral method are the following: 1° *How are various properties of the graph reflected in the spectrum of the graph?* 2° *Which properties of the graph can be determined by the spectrum of the graph and in which way?*

We quote some fundamental results in this area in order to give a more precise picture of the problems related to the questions posed.

Definition 1.5. *Two graphs are isospectral if they have the same spectra.*

Isomorphic graphs are simultaneously isospectral. However, isospectrality is not a sufficient condition for the isomorphism of graphs. The conjecture, which may be found and was posed by some mathematicians (F. Harary [34] and others), that isospectrality implies isomorphism of graphs, are disproved by a number of counterexamples. In Section 3.1. a list of known counterexamples is given.

Thus, a graph is not uniquely determined by its spectrum. Nevertheless, the spectrum yields several information about the graph. The information is higher when we restric ourselves to a narrower class of graphs. So, for example, for undirected graphs the spectral method is more valuable than for directed graphs. Let us quote an example confirming this assertion. All directed graphs without cycles have the spectrum containing only the numbers equal to zero, which is proved in **[82]**. With the same spectrum there is, in the set of undirected graphs, only one graph, i.e. the graph without edges.

A fundamental result in this area is due to H. SACHS **[78]**. From his paper we quote theorems on the basis of which it can be established which graphs have the same spectra. These results represent, in fact, specializations of results in connection with signal flow graphs (see **[34]**).

Theorem 1.1. *Let* $P_G(\lambda)=\sum_{i=0}^{n} a_i\,\lambda^{n-i}$ *be the characteristic polynomial of an arbitrary (directed) graph G. Then*

$$a_i=\sum_{L_i\subset G}(-1)^{p(L_i)}\quad (i=1,\dots,n),$$

where the summation is taken over all linear oriented subgraphs L_i *with exactly i vertices;* $p(L_i)$ *denotes the number of components of* L_i.

For undirected graphs we have the following specialization of Theorem 1.1.

Theorem 1.2. *Let* $P_G(\lambda)=\sum_{i=0}^{n} a_i\lambda^{n-i}$ *be the characteristic polynomial of an undirected graph. Elementary figures are: a) a graph with two vertices joined by an edge, b) cycle with* $p\,(p\geqslant 1)$ *vertices. The basic figure* U_i *is every graph, whose all components are elementary figures. Let the figure* U_i *have* $p(U_i)$ *components out of which* $c(U_i)$ *are cycles. Then*

$$a_i=\sum_{U_i\subset G}(-1)^{p(U_i)}2^{c(U_i)},$$

where the summation is taken over all basic figures U_i *with exactly i vertices which, as subgraphs, are contained in G.*

These results represent a generalization of those from **[10]**. Equivalent results were given later by other authors **[71]**, **[91]**, **[95]**. Except for this, it is shown in **[91]** that not only the characteristic polynomial of the adjacency matrix, but also more general matrix functions do not determine the graph up to isomorphism.

The mentioned results of H. SACHS, though they contain, in principle, the answers to the questions posed on page 4, cannot automatically solve all concrete problems which can be posed in connection with a given graph. The reason is that the linear subgraphs or basic figures are not in a simple connection with those characteristics of graphs which are interesting in applications. Therefore, further investigations are necessary in which, naturally, Theorems 1.1 and 1.2 are very significant.

Except the characteristic polynomial of the adjacency matrix of the graph some other functions of the adjacency matrix are studied in the literature (see, for example, [91]). Some authors have investigated the characteristic polynomial and the spectrum of other matrices corresponding to the graph. There are three groups of papers with this topic:

1. In papers [96], [103], [104], [105] the matrix $C=A+D$, where A is the adjacency matrix and $D=\|\delta_{ij}d_i\|_1^n$ (δ_{ij} KRONECKER' s δ-symbol, d_i the degree of the vertex x_i) the matrix of vertex-degrees, as well as its characteristic polynomial $P_C(\lambda)$ are treated. The matrix C is called *матрица соседства* in order to differ from the matrix A named *матрица смежности* in Soviet literature.

2. A. K. KEL'MANS ([98] — [101]) investigated a graph function related to the characteristic polynomial of the matrix $D-A$. For the graph G with n vertices he introduced the funtion $B_\lambda^n(G)=\frac{1}{\lambda}\,det\,(D-A+\lambda I)$.

3. In papers of J. SEIDEL and others ([33], [58], [83], [84], [85]) the $(-1, 1, 0)$ - adjacency matrix and its characteristic polynomial $P_B(\lambda)$ are used for undirected graph without loops or multiple edges. The mentioned matrix is of the form $B=\|b_{ij}\|_1^n$, where n is the number of vertices, $b_{ii}=0$, $b_{ij}=-1$ if x_i and x_j are adjacent vertices and $b_{ij}=1$ in the opposite case. This matrix is connected with the adjacency matrix by the relation $B=J-2A-I$, where J denotes the matrix whose all entries are equal to 1.

If G is a regular graph, all the mentioned functions can be brought in a simple algebraic connection with the characteristic polynomial of the adjacency matrix and the properties of these functions are easily transferred to the mentioned polynomial. In the case of nonregular graphs each of these three functions has its own properties. We shall use the results related to these functions only for regular graphs. Connections of mentioned functions with characteristic polynomial $P_G(\lambda)$ of a regular graph G of degree r are given by relations

$$P_C(\lambda)=P_G(\lambda-r), \tag{1.1}$$

$$B_\lambda^n(G)=\frac{1}{\lambda}P_G(\lambda+r), \tag{1.2}$$

$$P_B(\lambda)=(-1)^n\,2^n\,\frac{\lambda+1+2r-n}{\lambda+2r+1}\,P_G\left(-\frac{\lambda+1}{2}\right). \tag{1.3}$$

The first and the second relation are obvious. The third relation can be obtained, for example, in the way in which the connection between the characteristic polynomials of the graph and its complement in [77] is obtained.

Note that the analysis of graphs can be made by various matrices corresponded to the graph, ignoring characteristic polynomials or spectra. There is an abundant literature dealing with the above subject.

Several theorems of matrix theory have interpretations in the graph theory. Note that during the last ten years several authors have dealt with the inverse problem, i.e., with the use of graph theory for obtaining results in the matrix theory (see, for example, [27]). Some results from these papers can be of interest for the spectral method.

Finally, note that all mentioned kinds of spectra have a linear base. In [69] some non-linear spectra are defined and their investigation is suggested.

1.1. Fundamental properties of the spectrum of a graph

Some fundamental properties of spectra of graphs can be immediately established using several theorems of matrix theory.

Definition 1.6. *A matrix is called non-negative if all its elements are non-negative numbers.*

Since the adjacency matrix of a graph is non-negative, the spectrum of the graph has the properties of the spectrum of non-negative matrices. For non-negative matrices the following theorem holds (see, for example, [**32**], vol. II, p. 66):

Theorem 1.3. *A non-negative matrix always has a non-negative eigenvalue r such that the moduli of all its eigenvalues do not exceed r. To this „maximal" eigenvalue an eigenvector with non-negative coordinates corresponds.*

In the further text a vector with positive (non-negative) coordinates will be called a positive (non-negative) vector.

Definition 1.7. *A matrix A is called reducible if there is a permutation matrix P such that the matrix $P^{-1}AP$ is of the form $\begin{Vmatrix} X & O \\ Y & Z \end{Vmatrix}$, where X and Z are square matrices. Otherwise, A is called irreducible.*

Spectral properties of irreducible non-negative matrices are described by the following theorem of Frobenius ([**32**], vol. II, p. 53—54).

Theorem 1.4. *An irreducible non-negative matrix A always has a positive eigenvalue r that is a simple root of the characteristic polynomial. The moduli of all the other eigenvalues do not exceed r. To the "maximal" eigenvalue r there corresponds a positive eigenvector. Moreover, if A has h eigenvalues of modulus r, then these numbers are all distinct and are roots of the equation $\lambda^h - r^h = 0$. More generally: the whole spectrum $\{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ of A, regarded as a system of points in the complex λ - plane, goes over into itself under a rotation of the plane by the angle $\frac{2\pi}{h}$. If $h > 1$, then A can be put, by means of a permutation of rows and by the same permutation of columns, into the following "cyclic" form*

$$A = \begin{Vmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & A_{h-1,h} \\ A_{h1} & 0 & 0 & & 0 \end{Vmatrix},$$

where there are square blocks along the main diagonal.

According to Theorem 1.3 the spectrum of a graph lies in the circle $|\lambda| \leqq r$, where r is the greatest real eigenvalue. This eigenvalue is called the index of the graph. The algebraic multiplicity of the index can be greater than 1 and the corresponding eigenvector is, in general, non-negative.

It is known that irreducibility of the adjacency matrix is related to the property of connectedness of the graph. To a strongly connected graph an irreducible adjacency matrix corresonds and a graph with irreducible adjacency matrix has a property of the strong connectedness **[27]**, **[82]**. The strong connectedness is reduced in undirected graphs to the connectedness.

According to Theorem 1.4, the index of a strongly connected graph is a simple eigenvalue of the adjacency matrix and a positive eigenvector belongs to it.

For the undirected graph G, whose components are graphs $G_1, \ldots, G_S$ with adjacency matrices $A_1, \ldots, A_S$ the adjacency matrix A is a direct sum of $A_1, \ldots, A_S$. It can easily be seen that the following relation holds:

$$P_G(\lambda) = P_{G_1}(\lambda) \cdots P_{G_S}(\lambda). \tag{1.4}$$

Hence, the spectrum of the graph is the union of the spectra of its components, where the attention should be paid to algebraic multiplicity of particular eigenvalues. In **[82]** it is proved that (1.4) holds for an arbitrary graph, where $G_1, \ldots, G_S$ represent the components of the strong connectedness of G.

We shall now list some more theorems of the matrix theory showing new spectral properties of graphs.

Theorem 1.5. (See for example, **[32]** vol II, p. 69) *The "maximal" eigenvalue r' of every principal submatrix (of order less than n) of a non-negative matrix A does not exceed the "maximal" eigenvalue r of A. If A is irreducible, then always $r' < r$ holds. If A is reducible, then $r' = r$ holds for at least one principal submatrix.*

Theorem 1.6. (See, for example, **[10]**) *The increase of any element of non--negative matrix A does not decrease the "maximal" eigenvalue. The "maximal" eigenvalue increases strictly if A is an irreducible matrix.*

Theorem 1.7. (See, for example, **[59]** p. 64) *All the eigenvalues of a hermitian are real numbers.*

Theorem 1.8. (See, for example, **[37]**) *Let A be a real symmetric matrix, whose greatest and smallest eigenvalues are denoted by r and q. Let x be the eigenvector belonging to r. For a principal submatrix B of the matrix A let q' be the smallest eigenvalue whose eigenvector is denoted by y. Then $q' \geqq q$. If $q' = q$, vector y is orthogonal to the projection of vector x on the subspace corresponding to B.*

Theorem 1.9. (See, for example, **[59]** p. 119) *Let A be a hermitian with eigenvalues $\lambda_1, \ldots, \lambda_n$ $(\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n)$ and B one of its principal submatrices; B has the eigenvalues $\mu_1, \ldots, \mu_m$ $(\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_m)$. Then, the inequalities $\lambda_{n-m+i} \leqq \mu_i \leqq \lambda_i$ $(i = 1, \ldots, m)$ hold.*

Theorem 1.10. (See, for example, **[32]** vol. II, p. 79) *If the "maximal" eigenvalue r of a non-negative matrix A is simple and if positive characteristic vectors belong to r both in A and A^T, then A is irreducible.*

Theorem 1.11. (See, for example, **[32]** vol. II, p. 78) *To the "maximal" eigenvalue r of a non-negative matrix A there belongs a positive eigenvector both in A and A^T if and only if A can be represented by a permutation of rows and by the same permutation of columns in quasi-diagonal form $A = diag(A_1, \ldots, A_s)$, where $A_1 \ldots, A_s$ are irreducible matrices each of which has r as its "maximal" eigenvalue.*

Theorems 1.5 and 1.6 state that in strongly connected graphs every subgraph has the index smaller than the index of the graph.

The adjacency matrix of an undirected graph is symmetric (i. e., hermitian) and the spectrum of the graph, containing only real numbers, according to Theorem 1.7, lies in the segment $[-r, r]$.

Let $\{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of a graph. The number of loops is equal to the trace of the adjacency matrix. Therefore, we have for the graphs without loops $tr A = 0$, i. e., $\lambda_1 + \cdots + \lambda_n = 0$. The number of vertices of the graph is, naturally, equal to n, and for undirected graphs without loops or multiple edges the number of edges is equal to $m = \frac{1}{2} \sum_{i=1}^{n} \lambda_i^2$ (see **4.1.**). Hence, for this class of the graphs the basic element of the graph can be immediately determined from the spectrum.

It is quoted in **[10]** that for the index r of an undirected graph without loops or multiple edges the inequality $2 \cos \frac{\pi}{n+1} \leqq r \leqq n-1$ holds, where n denotes the number of vertices. The lower bound is reached for the trees having only two vertices of degree 1, and the upper bound for complete graphs. If we omit the assumption of connectedness, according to the foregoing, for a graph without edges we have, $r = 0$, and otherwise $r \geqq 1$.

For the smallest eigenvalue q from the spectrum of the graph G the inequality $-r \leqq q \leqq 0$ holds. For the graph without edges we have $q = 0$. Otherwise $q \leqq -1$. This is a consequence of Theorem 1.8. Namely, if q were greater than -1 we should have that a principal submatrix of the adjacency matrix has the smallest eigenvalue which is less than q. This submatrix corresponds to the graph $K_{1,1}$, having two vertices and one edge joining these two vertices, which surely exists as an induced subgraph in G, because G contains at least an edge. We have $q = -1$ if and only if all components of G are complete graphs (Theorem 3.4). The lower bound $q = -r$ is achieved, if the component of G, having the greatest index, represents a bipartite (bichromatic) graph (Theorem 4.3).

According to the foregoing, the following theorem precising the fundamental spectral properties of undirected graphs without loops or multiple edges, can be formulated.

Theorem 1.12. *For the spectrum $\{\lambda_1, \ldots, \lambda_n\}$ of an undirected graph G without loops or multiple edges the following statements hold:*

1° *The numbers $\lambda_1, \ldots, \lambda_n$ are real and $\lambda_1 + \cdots + \lambda_n = 0$;*

2° *If G contains no edges, we have $\lambda_1 = \cdots = \lambda_n = 0$;*

3° *If G contains at least one edge the below stated inequalities hold for the greatest number $\lambda_1 = r$ and for the smallest $\lambda_n = q$, from the spectrum:*

$$1 \leqq r \leqq n-1, \tag{1.5}$$

$$-r \leqq q \leqq -1. \tag{1.6}$$

In (1.5) *the upper bound is attained if and only if G is a complete graph, while the lower bound is reached if and only if G has, as components, graphs $K_{1,1}$ or isolated vertices, where at least one $K_{1,1}$ must exist. In* (1.6) *the upper bound is reached if G contains, as components, complete graphs, and the lower bound if and only if the component of G, having the greatest index, is a bipartite graph.*

If G is a connected graph, the lower bound in (1.5) *is replaced with* $2\cos\frac{\pi}{n+1}$. *Then equality holds if and only if G is a tree with two vertices of degree* 1.

We shall now list some spectral characteristics of regular graphs.

The index of a regular graph is equal to the degree of the graph [**10**]. It can easily be seen that this holds for unconnected graphs too, but then the index is not a simple eigenvalue. The multiplicity of the index is equal to the number of components. Immediately it can be proved that an eigenvector, having all coordinates equal to 1, corresponds to the index of a regular graph. In a connected graph the eigenvectors of other eigenvalues are orthogonal to this vector, i. e., the sum of their coordinates is equal to 0.

Further spectral properties of graphs can be obtained starting from the fact that the coefficients of the characteristical polynomials are integers. It follows from this that the elementary symmetric functions and sums of k-th powers (k a natural number) of eigenvalues are integers too. Since the coefficient of the oldest term of the characteristic polynomial is equal to 1, rational eigenvalues (if they exist) are integers.

Eigenvalues and eigenvectors of the adjacency matrix of a graph have also the following property related to the structure of the graph. Let λ be an eigenvalue and $u=(u_1, \ldots, u_n)$ the corresponding eigenvector of the adjacency matrix A of the graph G. We shall correspond to the vertex x_i the quantity u_i $(i=1, \ldots, n)$. If the equality $Au=\lambda u$ is written in the scalar form, we have, for every i, $\lambda u_i=\sum_j a_{ij} u_j$, where summation is made over indices j of those vertices x_j in which at least an edge from the vertex x_i leads. For undirected graphs without loops or multiple edges it can be written $\lambda u_i=\sum_j u_j$. In this case for every vertex x_i the sum of quantities u_j of vertex x_j, adjacent to x_i, is λ times greater than the quantity u_i of the vertex x_i.

Despite several conditions which must be satisfied by a graph spectrum, we are not familiar with any procedure, except a direct verification, by which we can establish whether or not a given set of numbers represents the spectrum of a graph.

2. ON SOME COMBINATORIAL PROBLEMS RELATED TO SPECTRA OF GRAPHS

In this Chapter we start from some combinatorial problems related to spectra of graphs. One combinatorial model (variations with restrictions), suitable for solving various combinatorial problems, is described. The number of variations with restriction is connected with the number of walks[1)] in a graph and corresponding generating function is determined.

In **2.2.** it is shown, that this generating function can for some graphs be determined by generating functions of simpler graphs.

In **2.3.** it is established that the obtained results represent the basis for the study of several spectral properties of graphs. The notion of the main part of the spectrum of the graph, which is efficiently applied to proving certain theorems, is also introduced.

2.1. The number of variations with restrictions and the number of walks in a graph

Variation of the k-th class (also called: permutation k at a time) with repetitions of the set $X=\{x_1, \ldots, x_n\}$ is every ordered k-tuple $(x_{i_1}, \ldots, x_{i_k})$ where $i_j \in \{1, \ldots, n\}$ $(j=1, \ldots, k)$. The number of such variations is $\overline{V}_n^k=n^k$.

[1)] In my previous papers the word "path" was used.

During the formation of variations with repetitions it is possible to impose certain restrictions. In this paper we shall consider variations with restrictions of the following type.

Definition 2.1. *Let the set $X=\{x_1, \ldots, x_n\}$ be given. We say that on X restrictions are defined if for every x_i the set X is decomposed into two disjoint sets X_{i1} and X_{i2}. Permitted variation with repetitions of the set X under given restrictions is every k-tuple $(x_{i_1}, \ldots, x_{i_k})$ in which after its arbitrary element x_i, which is not the last in that variation, appears an element from the set X_{i1}.*

Definition 2.2. *A pair (x_i, x_j) of adjacent elements of a variation is called a permitted pair if and only if $x_j \in X_{i1}$. The square matrix $A=\| a_{ij} \|_1^n$, where $a_{ij}=1$ if x_i, x_j is a permitted pair and $a_{ij}=0$ otherwise, is called the matrix of permitted pairs. The matrix of restrictions $\overline{A}$ is obtained from A by interchanging 0 and 1.*

In [17] the number $\overline{V}_n^k(A)$ of variations with repetition of the k-th class of a set with n elements, with a given matrix of permitted pairs is determined.

We shall connect this problem with the problem of determining the number of walks in a graph. (Under "walk of length k" we understand a sequence $u_1, \ldots, u_k$ of the oriented edges of the graph, where it is not necessary that $u_i \neq u_j$ and where for $i=2, 3, \ldots, k$ the edge u_i starts from that vertex in which u_{i-1} terminates. An edge can be a loop. In case of undirected graphs every edge is to be replaced with two oriented edges with mutually opposed orientations.)

If A is interpreted as the adjacency matrix of graph G with vertices $x_1, \ldots, x_n$, it can easily be seen that the number $\overline{V}_n^k(A)$ is equal to the number of all walks of length $k-1$ in G.

The starting point for considerations in [17] was the following well known theorem (see, for example, [3], p. 124):

Theorem 2.1. *If A is the adjacency matrix of the arbitrary graph G, the element at the place (i,j) of the matrix A^k $(k=0, 1, 2, \ldots)$ is equal to the number of walks of length k, leading from the vertex x_i to the vertex x_j.*

In [17] the following theorem is proved.

Theorem 2.2. *The function*

$$G(t)=(-1)^n \frac{P_{\overline{A}}\left(-\frac{1}{t}\right)}{P_A\left(\frac{1}{t}\right)} \tag{2.1}$$

is generating function for the numbers $\overline{V}_n^k(A)$ of the variations of the k-th class with repetitions and with the matrix of permitted pairs A, where the following relation holds

$$\overline{V}_n^k(A)=\frac{1}{k!} G^{(k)}(0) \qquad (k=1, 2, \ldots). \tag{2.2}$$

For $k=0$ one obtains $\overline{V}_n^0(A)=1$.

From (2.1) and (2.2) it follows that the generating function $H_G(t)=\sum_{k=0}^{+\infty} N_k t^k$ for the numbers N_k of the walks of the lengths k, of the graph G, with the adjacency matrix A, is given by the expression

$$H_G(t)=\frac{1}{t}\left[(-1)^n\frac{P_{\bar{A}}\left(-\frac{1}{t}\right)}{P_A\left(\frac{1}{t}\right)}-1\right],$$

where $\bar{A}$ is to be interpreted as $\bar{A}=J-A$; J representing a square matrix whose all elements are equal to 1.

However, if G has multiple edges or multiple loops, then the matrix $\bar{A}$ has no significance. If the maximum number of edges between the two vertices or loops of a vertex in G is equal to p (i.e. if G is a p-graph), it is convenient to express the generating function by characteristic polynomial of the matrix $\bar{A}^p=pJ-A$, which can be interpreted as the adjacency matrix of the complement of G. Using derivations similar to those of **[17]** we arrive at the following theorem.

Theorem 2.3. *Let G be a p-graph with the adjacency matrix A. The generating function for the numbers of walks in G is given by*

$$H_G(t)=\frac{1}{pt}\left[(-1)^n\frac{P_{\bar{A}^p}\left(-\frac{1}{t}\right)}{P_A\left(\frac{1}{t}\right)}-1\right].$$

The following theorem is also proved in **[17]**.

Theorem 2.4. *The generating function for the numbers of walks of the undirected graph G without loops or multiple edges is given by*

$$H_G(t)=\frac{1}{t}\left[(-1)^n\frac{P_{\bar{G}}\left(-\frac{t+1}{t}\right)}{P_G\left(\frac{1}{t}\right)}-1\right]. \tag{2.3}$$

In **[53]** P. W. Casteleyn gives the expression for the generating function for numbers of walks between two prescribed vertices of graph, but in that expression the characteristic polynomial of the complementary graph does not appear.

Example 2.1. A regular graph of the degree r, with n vertices, has, obviously $N_k=nr^k$ walks of the lengths k, and therefore

$$H_G(t)=\sum_{k=0}^{+\infty} nr^k t^k=\frac{n}{1-rt}.$$

For $r=n-1$ one obtains the complete graphs and for $r=0$ the graph, which contains only isolated vertices. The graph, which has only one vertex without edges and loops, has the generating function of the form $H_G(t)=1$.

Example 2.2. In [63] the combinations of the set $\{1,\ldots,n\}$ with restricted differences and cospan were considered. For the combination of the p-th class $\{x_1,\ldots,x_p\}$, $1\leq x_1<\cdots<x_p\leq n$, differences are defined by $d_j=x_{j+1}-x_j$, $j=1,\ldots,p-1$, and span by

$d = x_p - x_1$. The number of combinations, satisfying restrictions $k \leqq d_j \leqq k'$, $j = 1, \ldots, p-1$, $l \leqq n-d \leqq l'$, is determined. Special cases of these combinations with a generalizatión in another direction can be dealt with by the use of the above proposed procedure. Consider combinations of the p-th class with repetitions for which we have $d_j \in M$ $(j = 1, \ldots, p)$, $M \subset \{1, \ldots, n\}$ and for which the greatest element m from M satisfies the condition $(p-1)\,m < 2n$. The number of such combinations can be determined by considering the graph G, having vertices $x_1, \ldots, x_n$ and in which, for every pair of indices (i, j), a (directed) edge leads from the vertex x_i to the vertex x_j if and only if $(j-i) \pmod n \in M$. Besides, G has a loop at every vertex. Exactly p closed walks of length p correspond to each of the considered combinations. Only those closed walks, which, due to the existence of loops, do not leave the vertex from which they start, correspond to none of the combinations. The number of such closed walks is equal to n. According to theorem 2.1 the number of closed walks of length k is equal to the trace of the k-th power of the adjacency matrix. If $\{\lambda_1, \ldots, \lambda_n\}$ is the spectrum of the graph G, the number of considered combinations, except of some trivial cases, is equal to $\frac{1}{p} \sum_{i=1}^{n} \lambda_i^p - n$.

In [89] M. Tuero determines the spectra of some graphs of described form.

Example 2.3. In how many ways N_{kn} the king (chessfigure) can make a series of k moves on a chessboard of dimensions $n \times n$?

The solution can be found in [18]. See also Example 5.3.

Example 2.4. To the determination of the number of walks in graph the following combinatorial problem, which is not still completely solved, can be reduced.

How many kings at most and in how many ways can they be placed on a chessboard of dimensions $m \times n$, so that they do not attack each other?

The problem is partially solved in [81]. The maximal number of kings for a square board of dimensions $n \times n$ is $\left(\frac{n+1}{2}\right)^2$ for n odd and $\left(\frac{n}{2}\right)^2$ for n even. In the first case there is only one way of placing the kings while in the second case the number is not determined in the general case. It is only known that any arrangement of kings must have the following properties.

Consider the chessboard with $2m \times 2n$ fields where $2m$ is the vertical and $2n$ the horizontal dimension, which represents a somewhat more general case. Let us partition this board by horizontal and vertical lines into squares containing four fields each. In each of these squares, on one of its four fields, one king can be placed. Thus, on the board mn kings in total can be placed. However, the kings cannot be arranged in an arbitrary way since the placing of a king on a square can prevent the placing of another king on some fields of the adjacent square. In order to determine the number of ways of placing of kings consider the first left vertical column of squares. It represents one rectangular part of the chessboard of dimensions $2m \times 2$, i. e. it contains m squares of dimensions 2×2. Put in this column m kings. It can easily be seen that the number of arrangements of these kings is $(m+1)\,2^m$. In the adjacent right column of squares the following m kings obviously cannot be placed in all $(m+1)\,2^m$ ways because some of arrangements are excluded by the very place of kings in the first column. Therefore, we shall define the graph G with $(m+1)\,2^m$ vertices corresponding to the mentioned arrangements of kings in one column of squares. From the vertex x an oriented edge leads to the vertex y, if and only if, after the column of squares, corresponding to the vertex x, the column, corresponding to the vertex y, is to be found.

The number of arrangements of mn kings on the $2m \times 2n$ board is equal to the number of walks of length $n-1$ in the graph G.

Consider the case $m = 2$. Every column contains two squares. Let us number the fields in every square by 1, 2, 3, 4 according to the following scheme

$$\begin{matrix} 1 & 2 \\ 3 & 4. \end{matrix}$$

Then in the first left column of the board the following 12 arrangements of kings can appear:

$$\begin{matrix} 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 1 & 2 & 1 & 2 & 3 & 4 & 3 & 4 & 3 & 4 & 3 & 4. \end{matrix}$$

The adjacency matrix of this graph is

$$A=\left\|\begin{array}{cccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right\|.$$

It seems possible to solve this problem, in general case, by means of the above described method.

2.2. Generating function for the numbers of walks and characteristic polynomial of the complete product of graphs

In this Section we shall consider only undirected graphs without loops or multiple edges. $\overline{G}$ denotes the graph complementary to the graph G, and G' the graph, which can be obtained, if to each of the vertices of G one loop is added. The adjacency matrix of the graph $\overline{G}$ is $\overline{A}-I$, and of G' is $A+I$.

We consider also the following type of sum and product of graphs.

Definition 2.3. *The direct sum $G_1 \dot{+} G_2$ of graphs G_1 and G_2 is a graph containing all the vertices and all the edges of boths graphs G_1 and G_2, and no other vertices or edges. The graph $G_1 \nabla G_2$ is called the complete product of graphs G_1 and G_2 and is obtainable from $G_1 \dot{+} G_2$ if each of the vertices of G_1 is joined by an edge with each of the vertices of G_2.*

In [17] the following theorem is proved:

Theorem 2.5. *For the generating function $H_G(t)$ for the numbers of walks of the graph G the following formulas hold:*

$$H_{G'}(t)=\frac{1}{1-t}\,H_G\left(\frac{1}{1-t}\right), \tag{2.4}$$

$$H_{\overline{G}}(t)=\frac{H_G\left(-\dfrac{t}{t+1}\right)}{t+1-t\,H_G\left(-\dfrac{t}{t+1}\right)}, \tag{2.5}$$

$$H_{G_1 \dot{+} G_2}(t)=H_{G_1}(t)+H_{G_2}(t), \tag{2.6}$$

$$H_{G_1 \nabla G_2}(t)=\frac{H_{G_1}(t)+H_{G_2}(t)+2t\,H_{G_1}(t)\,H_{G_2}(t)}{1-t^2 H_{G_1}(t)\,H_{G_2}(t)}. \tag{2.7}$$

EXAMPLE 2.5. The bicomplete graph K_{n_1,n_2} can be represented as the ∇-product of graphs G_1 and G_2, both of which contain only isolated vertices. We then have $H_{G_1}(t)=n_1$ and $H_{G_2}(t)=n_2$, and according to (2.7), for the bicomplete graph we have

$$H_{K_{n_1,n_2}}(t)=\frac{n_1+n_2+2n_1n_2t}{1-n_1n_2t^2}.$$

Specially, for $n_1=n$ and $n_2=1$ the considered graph represents a star and corresponding generating function is

$$H_{K_{n,1}}(t)=\frac{n+1+2nt}{1-nt^2}.$$

EXAMPLE 2.6. Determine the generating function for the numbers of walks for the graph obtained by deleting m nonadjacent edges from a complete graph with n vertices. The complement of this graph contains, as components, m regular connected graphs of degree 1 and $n-2m$ isolated vertices. Hence,

$$H_{\bar{G}}(t)=m\frac{2}{1-t}+n-2m=\frac{n-t(n-2m)}{1-t}$$

and, according to (2.5),

$$H_G(t)=\frac{n+2t(n-m)}{1-t(n-3)-2(n-m-1)t^2}.$$

EXAMPLE 2.7. If the function $H_G(t)$ is known, the connection between characteristic polynomials of graph and its complement can be established. This fact can be used to determine the characteristic polynomials of some graphs.

A graph is k-complete if the set of its vertices can be partitioned into k groups in such a way that every two vertices from different groups are adjacent and every two vertices from the same group are not adjacent. If, then, these k groups have $n_1,\dots,n_k$ vertices respectively, the graph is denoted by $K_{n_1,\dots,n_k}$.

First we have

$$H_{\bar{K}_{n_1,\dots,n_k}}(t)=\sum_{i=1}^{k}\frac{n_i}{1-(n_i-1)t},$$

because $\bar{K}_{n_1,\dots,n_k}$ is a direct sum of complete graphs with $n_1,\dots,n_k$ vertices respectively. According to (2.5) we have

$$H_{K_{n_1,\dots,n_k}}(t)=\frac{\sum_{i=1}^{k}\frac{n_i}{1+n_it}}{1-t\sum_{i=1}^{k}\frac{n_i}{1+n_it}}.$$

On the other hand, we have, according to (1.4) and to the Example 2.9.

$$P_{\bar{K}_{n_1,\dots,n_k}}(t)=(t+1)^{n-k}\prod_{i=1}^{k}(t-n_i+1).$$

Formula (2.3) can be written in the form

$$P_G(t)=(-1)^n\frac{tP_{\bar{G}}(-t-1)}{H_G\left(\frac{1}{t}\right)+t}.$$

According to foregoing, we obtain

$$P_{K_{n_1,\ldots,n_k}}(t)=t^{n-k}\left(1-\sum_{i=1}^{k}\frac{n_i}{t+n_i}\right)\prod_{i=1}^{k}(t+n_i).$$

If $k=2$, we get the case from the Example 2.8.

If $n_1=\cdots=n_k=\frac{n}{k}$, a result from **[30]** is obtained.

Theorem 2.5 enables the determination of the characteristic polynomial of a complete product of graphs. In virtue of (2.3) and (1.4), (2.6) becomes

$$\frac{1}{t}\left[(-1)^{n_1+n_2}\frac{P_{\overline{G_1\dotplus G_2}}\left(-\frac{1}{t}-1\right)}{P_{G_1}\left(\frac{1}{t}\right)P_{G_2}\left(\frac{1}{t}\right)}-1\right]=\sum_{i=1}^{2}\frac{1}{t}\left[(-1)^{n_i}\frac{P_{\overline{G}_i}\left(-\frac{1}{t}-1\right)}{P_{G_i}\left(\frac{1}{t}\right)}-1\right].$$

Since $\overline{G_1\dotplus G_2}=\overline{G}_1\nabla\overline{G}_2$, putting $-\frac{1}{t}-1=\lambda$ and substituting $\overline{G}_1$, $\overline{G}_2$ for G_1, G_2, i. e. G_1, G_2 for $\overline{G}_1$, $\overline{G}_2$, the following theorem is arrived at:

Theorem 2.6. *The characteristic polynomial of the ∇-product of graphs is given by the relation*

(2.8)
$$P_{G_1\nabla G_2}(\lambda)=(-1)^{n_2}P_{G_1}(\lambda)\,P_{\overline{G}_2}(-\lambda-1)$$
$$+(-1)^{n_1}P_{G_2}(\lambda)\,P_{\overline{G}_1}(-\lambda-1)-(-1)^{n_1+n_2}P_{\overline{G}_1}(-\lambda-1)\,P_{\overline{G}_2}(-\lambda-1).$$

If G_1 and G_2 are regular graphs, Theorem 2.6 together with Theorem 2.10. gives the following result from **[29]**:

Theorem 2.7. *The characteristic polynomial of the complete product of regular graphs G_1 and G_2 is given by the ralation:*

(2.9)
$$P_{G_1\nabla G_2}(\lambda)=\frac{P_{G_1}(\lambda)P_{G_2}(\lambda)}{(\lambda-r_1)(\lambda-r_2)}[(\lambda-r_1)(\lambda-r_2)-n_1n_2].$$

EXAMPLE 2.8. (FINCK and GROHMANN **[29]**). The relation $K_{n_1,n_2}=G_1\nabla G_2$ holds for the bicomplete graph K_{n_1,n_2}, where G_1 and G_2 are graphs consisting of n_1 respectively n_2 isolated vertices. Obviously $P_{G_1}(\lambda)=\lambda^{n_1}$ and $P_{G_2}(\lambda)=\lambda^{n_2}$. On the basis of (2.9) we then have $P_{K_{n_1,n_2}}(\lambda)=(\lambda^2-n_1n_2)\lambda^{n_1+n_2-2}$.

2.3. The main part of the spectrum

Let $\{\lambda_1,\ldots,\lambda_n\}$ be the spectrum of an undirected graph without loops or multiple edges. In **[17]** the following definition of the main part of the spectrum is given.

Definition 2.4. *The main part of the spectrum of a graph is the set of all these eigenvalues λ_i, mutually different, for which, in the expression*

(2.10)
$$N_k=C_1\lambda_1{}^k+\cdots+C_m\lambda_m{}^k$$

for the number of walks, $C_i\neq 0$ holds.

Hence, the main part of the spectrum is a subset of the zeros of the minimal polynomial of the adjacency matrix of the graph. In regular graphs the main part of the spectrum contains only the index of the graph and, as it shall be seen (Theorem 2.13), only the regular graphs have this property.

We shall define one notion more, which enables research of some properties of the main part of the spectrum.

Regular graphs have the property that the same number of walks of certain length k starts from the every vertex (compare with Theorem 4.16 and with the result of T. H. WEI). If r denotes the degree of the graph, the number of these walks is r^k, i.e. the total number of walks is, as it has already been said, $N_k = nr^k$, where n is the number of vertices. Therefrom we obtain $r = \sqrt[k]{\frac{N_k}{n}}$. We see that the expression $\sqrt[k]{\frac{N_k}{n}}$ defines a certain kind of a mean value of degrees of vertices. In regular graphs the expression $\sqrt[k]{\frac{N_k}{n}}$ does not depend on k, since all the degrees of vertices are mutually equal.

Definition 2.5. *The dynamical mean value d of degrees of vertices in a graph is given by* $d = \lim\limits_{k \to +\infty} \sqrt[k]{\frac{N_k}{n}}$.

Theorem 2.8. *The dynamical mean value of the degrees of vertices is equal to the index of the graph.*

Proof. We shall primarily prove a lemma.

Lemma 2.1. *If $\lambda_1 = r$ (r being the index of the graph), and if the graph contains at least one edge, the inequality $C_1 > 0$, holds for the quantity C_1 from* (2.10).

Proof of the Lemma. We see from [**17**] that each element of A^k is of the form (2.10), where, naturally, the coefficients C_i depend on the ordinal numbers of the row and of the column of the considered element. The coefficient of $\lambda_1{}^k$ is not negative, since in the opposite case, for sufficiently great k, the considered element would be negative. If we add up the corresponding expressions for all the elements of A^k, we obtain (2.10), and it must be $C_1 \geqq 0$. However, $C_1 = 0$ is impossible, since $\operatorname{tr} A^k = \lambda_1{}^k + \cdots + \lambda_n{}^k$, and this implies that the smallest value of C_1 is equal to 1.

With this Lemma 2.1 is proved.

From the Lemma and according to the definition of the dynamical mean value of the degrees of vertices the statement of Theorem 2.8 follows.

The concept of the main part of the spectrum of a graph facilitate the proofs of a number of theorems, whose descriptions are given in the further text.

The arguments, used in the proof of Theorem 3 from [**17**], are sufficient for proving a more general theorem. Actually, the generating function for the number of walks is of the form

$$H_G(t) = u\left[\frac{(u+\bar{\lambda}_1+1)\cdots(u+\bar{\lambda}_n+1)}{(u-\lambda_1)\cdots(u-\lambda_n)} - 1\right],$$

where $u=\frac{1}{t}$ and where $\{\overline{\lambda}_1, \ldots, \overline{\lambda}_n\}$ denotes the spectrum of the graph $\overline{G}$. Then we have

$$(2.11)\qquad \psi(u) \overset{\text{def}}{=} \frac{(u+\overline{\lambda}_1+1)\cdots(u+\overline{\lambda}_n+1)}{(u-\lambda_1)\cdots(u-\lambda_n)} = 1+\sum_{k=0}^{+\infty}\frac{N_k}{u^{k+1}} = 1+\sum_{i=1}^{m}\frac{C_i}{u-\lambda_i}.$$

We see that $\psi(u)$ must be a rational function having only simple poles. The set of these poles is equal to the main part of the spectrum of the graph. Thus, multiple factors in the denominator of expression (2.11) have to be cancelled, so that, after all possible cancellations, the zeros of the polynomial in the denominator represent the main part of the spectrum. So we have the following theorem:

Theorem 2.9. *If the spectrum of G contains eigenvalue λ with multiplicity $p\ (p>1)$, then the spectrum of the complement $\overline{G}$ contains eigenvalue $-\lambda-1$, whose multiplicity $\overline{p}$ satisfies the inequality $\overline{p}\geqq p-1$.*

This theorem has several interesting corollaries. Corollary 2 is published in [17] as Theorem 3.

Corollary 1. *If Theorem 2.9 is applied to the complement $\overline{G}$ of G, the inequality $p\geqq\overline{p}-1$ is obtained, which together with the inequality from the mentioned theorem gives $p-1\leqq\overline{p}\leqq p+1$.*

Corollary 2. *Let G be a self-complementary graph. Then to each eigenvalue λ_i from the spectrum of G of the multiplicity $p_i\ (p_i>1)$ (if it exists) corresponds another eigenvalue λ_j, whose multiplicity p_j satisfies the inequality $p_i-1\leqq p_j\leqq p_i+1$, where $\lambda_i+\lambda_j=-1$.*

Corollary 3. *If the spectrum of the graph G_1 contains the eigenvalue λ with the multiplicity $p\ (p>2)$, the spectrum of the complete product $G_1\nabla G_2$ of the graph G_1 with the arbitrary graph G_2 contains λ as an eigenvalue of the multiplicity p', where $p'\geqq p-2$.*

Proof. Since $G_1\nabla G_2=\overline{\overline{G}_1\dot{+}\overline{G}_2}$, Corollary 3 follows immediately from Theorem 2.9 and the fact that the spectrum of direct sum of graphs is obtained by union of the spectra of graphs - summands.

Theorem 2.10. *Let G be a regular graph of degree r and with n vertices. To each of eigenvalues $\lambda_i\ (\lambda_i\neq r)$ from the spectrum of the multiplicity p_i the eigenvalue $-\lambda_i-1$, whose multiplicity is also p_i, corresponds in the spectrum of the complement $\overline{G}$, i. e. for characteristic polynomials of G and $\overline{G}$ the following relation holds:*

$$(2.12)\qquad P_{\overline{G}}(\lambda)=(-1)^n\frac{\lambda-n+r+1}{\lambda+r+1}P_G(-\lambda-1).$$

Proof. The main part of the spectrum of a regular graph contains only the index r of the graph. Therefore, all the factors, except $u-r$, from the denominator of expression (2.11) have to be cancelled. Since the index of the complement is equal to $n-1-r$, we obtain the statement of Theorem 2.10.

Corollary. *As it is already mentioned, Theorem* 2.10 *together with Theorem* 2.6 *yields Theorem* 2.7. *Also the statement of Corollary* 3 *of Theorem* 2.9 *can be precised. Namely, if* $p>1$, *it follows from* (2.8) *that* $P_{G_1 \nabla G_2}(\lambda)$ *has* λ *as a root of multiplicity* p', *where* $p' \geq p-1$.

EXAMPLE 2.9. Let G be a complete graph with n vertices. For the complement $\overline{G}$ obviously $P_{\overline{G}}(\lambda)=\lambda^n$ holds. According to (2.12) we have $P_G(\lambda)=(\lambda-n+1)(\lambda+1)^{n-1}$. If G is a regular graph of degree $n-2$, then $n=2k$, $P_{\overline{G}}(\lambda)=(\lambda^2-1)^k$ and $P_G(\lambda)=(\lambda-n+2)\lambda^k(\lambda+2)^{k-1}$.

Theorem 2.10 is proved in [77] in another way. The theorem given below is quoted in this paper as an immediate consequence of Theorem 2.10.

Theorem 2.11. *In a self-complementary regular graph, with* $n=4k+1$ *vertices, to every eigenvalue* λ_i $(\lambda_i \neq 2k)$ *another eigenvalue* λ_j, *where* $\lambda_i+\lambda_j=-1$, *corresponds.*

This result can be reached in a similar way as the previous theorem. It is interesting to compare this result with Corollary 2 of Theorem 2.9.

EXAMPLE 2.10. The question may be immediately asked, whether self-complementary graphs described in Corollary 2 and not involved by Theorem 2.11 exist? The self-complementary graphs with 4 or 5 vertices do not satisfy the required conditions. (As it is proved in [74] and [77] the self-complementary graphs have $4k$ or $4k+1$ vertices, where k is a nonnegative integer.) However, in [74] two self-complementary non-regular graphs with 8 vertices are quoted, whose adjacency matrices are:

$$A_1=\begin{Vmatrix} 0&1&1&1&1&0&1&0\\ 1&0&1&0&0&0&0&0\\ 1&1&0&1&1&0&1&0\\ 1&0&1&0&0&0&0&0\\ 1&0&1&0&0&1&1&1\\ 0&0&0&0&1&0&1&0\\ 1&0&1&0&1&1&0&1\\ 0&0&0&0&1&0&1&0 \end{Vmatrix}, \quad A_2=\begin{Vmatrix} 0&1&0&0&1&0&0&1\\ 1&0&1&0&1&0&0&1\\ 0&1&0&1&1&0&0&1\\ 0&0&1&0&1&0&0&1\\ 1&1&1&1&0&1&0&0\\ 0&0&0&0&1&0&1&0\\ 0&0&0&0&0&1&0&1\\ 1&1&1&1&0&0&1&0 \end{Vmatrix}.$$

The first graph has the spectrum $\left\{4, \frac{-1+\sqrt{17}}{2}, 0, 0, -1, -1, -1, \frac{-1-\sqrt{17}}{2}\right\}$. The spectrum of the second graph contains the numbers $\frac{\sqrt{5}-1}{2}$ and $-\frac{\sqrt{5}+1}{2}$ as two-fold eigenvalues. Hence the answer to the question asked is affirmative.

On the other hand, there are non-self-complementary graphs but having the property of self-complementary graphs expressed by Corollary 2. Such is, for example, the graph obtained from the complete pentagraph by deleting one edge. It has the spectrum $\{\sqrt{7}+1, 0, -1, -1, 1-\sqrt{7}\}$.

From the expression (2.11) the following theorem can be also immediately obtained:

Theorem 2.12. *The main part of the spectrum of the graph G and the main part of the spectrum of its complement* $\overline{G}$ *contain the same number of elements, where none eigenvalue* λ_i *from the main part of the spectrum of G and none eigenvalue* $\overline{\lambda}_j$ *from the main part of the spectrum of* $\overline{G}$ *satisfy the relation* $\lambda_i+\overline{\lambda}_j=-1$.

Using the generating function for the numbers of walks in the graph, the following theorem can be proved too:

Theorem 2.13. *Using abbrevations from Theorem* 2.10, *the relation* (2.12) *holds if and only if G is a regular graph.*

Primarily, we prove the following lemma:

Lemma 2.2. *The graph G is regular of degree r and has n vertices if and only if for the numbers N_k ($k=0, 1, 2, \ldots$) of walks of length k in G the relation $N_k = nr^k$ holds.*

Proof. If G is a regular graph of degree r with n vertices then, obviously, $N_k = nr^k$. If $N_k = nr^k$, then the number of vertices is equal to $N_0 = n$ and $N_1 = nr$. Since in every graph the number of walks of length 1 is equal to two-fold number of edges, we get $\bar{d} = \frac{2m}{n} = \frac{N_1}{n} = r$ for the mean value $\bar{d}$ of degrees of vertices. According to Theorem 4.10, the relation $\bar{d} = r$ implies that G is a regular graph.

***Proof of Theorem* 2.13.** It is sufficient to prove that the relation (2.12) implies the regularity of the graph. If (2.12) is inserted in the relation (2.3), $H_G(t) = \frac{n}{1-rt}$ i. e. $N_k = nr^k$ is obtained. According to Lemma 2.2, G is a regular graph.

This completes the proof of Theorem 2.13.

3. SPECTRAL CHARACTERISATIONS OF CERTAIN CLASSES OF GRAPHS

In this Chapter various variants of the following problem are considered:

The spectrum, i. e. some spectral characteristics of the graph are given. Let us determine all graphs from a given class of graphs having the given spectrum, i. e. the given spectral characteristics.

Thus, the possibility of graph identification as an entity is considered. In the following Chapter, as a contrast to this, the procedures enabling the determination of structural details of the graph on the basis of its spectrum are described.

In this Chapter we shall consider only the finite, undirected graphs without loops or multiple edges so that the term graph, if no particular mention is made, will be used for representatives of the above mentioned class of graphs.

3.1. The list of the familiar pairs of isospectral non-isomorphic graphs

In [10] L. Collatz and U. Sinogowitz have already noted that the spectrum of the graph does not determine the graph precisely up to isomorphism. In the mentioned paper the example of two isospectral trees with 8 vertices has been given. These trees have various sets of vertex degrees too.

The term „a pair of isospectral non-isomorphic graphs" will be denoted by PING. The literature points out to various examples of the PINGs. In this section we shall give some of the known examples of the PINGs. The importance of quoting these examples lies in the following:

1° For every pair of nonisomorphic graphs a set of characteristics that are different for the graphs from the pair, may be found. Therefore, every PING points out to the properties of graphs that are not uniquely determined by its spectrum.

2° Every PING proves the inaccuracy of a certain number of statements of the type B (see the following section).

We shall limit ourselves to the undirected graphs without loops or multiple edges. (It is relatively easy to construct PINGs for other kinds of graphs. Besides what has been said in **1.2.**, let us mention that in [**71**] the PING with directed graphs with 7 vertices has been cited. See also [**24**]).

F. Harary in [34] states that his conjecture, according to which isospectrality implies the isomorphism of graphs, has been disproved by R. C. Bose, who has given the PING with 16 vertices. According to [34], R. H. Bruck and A. J. Hoffman have also found the PINGs with 16 vertices.

In the set of connected graphs with utmost 5 vertices isospectral graphs do not exist which may be directly verified, for example, by a table of spectra of graphs from [**10**]. In [2] G. Baker gives the PING with connected graphs with 6 vertices which shows that the number 5 cannot be replaced in the above by a bigger number.

If we consider graphs without assumption of connectedness, then there exists the PING with 5 vertices. This pair forms the graphs whose adjacency matrices are:

$$A_1=\begin{Vmatrix}0&1&0&1&0\\1&0&1&0&0\\0&1&0&1&0\\1&0&1&0&0\\0&0&0&0&0\end{Vmatrix},\qquad A_2=\begin{Vmatrix}0&1&1&1&1\\1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\end{Vmatrix}.$$

This example can be generalized. The graph, having as components s isolated vertices and one bicomplete graph K_{n_1,n_2}, has, according to example 2.8, the spectrum containing numbers $\sqrt{n_1 n_2}$, $-\sqrt{n_1 n_2}$ and n_1+n_2-2+s numbers equal to 0. Consider the graph with the spectrum: $\sqrt{m}$, $-\sqrt{m}$ and $n-2$ numbers equal to 0 (m a natural number). This spectrum may belong to each of graphs of the above described type whose parameters n_1, n_2, s satisfy the equations $n_1+n_2+s=n$, $n_1 n_2=m$. These equations can have obviously several solutions in the set of natural numbers (s can be equal to 0). The unique solution exists if $n=k+l$ (k, l natural numbers, $kl=m$), where the quantity $|k-l|$ is the smallest possible. In this case $s=0$ and the spectrum determines a bicomplete graph. Thus, the bicomplete graph is determined by its spectrum if and only if the minimum of $|k-l|$ is reached for that pair (k, l) from $\{(k, l) \mid kl=n_1 n_2\}$ for which $k=n_1$ and $l=n_2$.

From these examples we see that the information on the connectedness of the graph is not, in general case, contained in the spectrum of the graph. If we consider any narrower class of graphs (for example, the class of regular graphs), the information on the connectedness can be attained from the spectrum (see Theorem 4.10). It is of interest that the function $B^n_\lambda(G)$, described in **1.1**, contains in general case the information on the connectedness (see [**100**], Theorem 6).

In [**91**] the PING with trees with 12 vertices is given. The mentioned trees have also the same sets of vertex degrees. The author expresses his pessimism in relation to the possibility of solving the graph isomorphism problem by spectra, even in the case when, apart from the spectrum, some additional information on the graph are supplied.

M. Fisher, who encountered the graph isospectrality problem at the investigation of membrane vibration problem [**31**], has considered graphs with the following restrictions: 1° graph does not contain a vertex of degree 1, 2° graph is planar, and some others. He has constructed an infinite sequence of PINGs satisfying conditions 1° and 2°. The PINGs from the sequence have $5n$ ($n=3, 4, \ldots$) vertices. It was established, however, that graphs satisfying condition 1° and having at most 6 vertices, are determined by their spectra.

An infinite sequence of sets of mutually non-isomorphic isospectral graphs is given by R. H. Bruck in [7].

It seems that the PINGs with graphs having a great number of vertices are an usual occurrence. Interesting data about the statistic of the PINGs are given by G. Baker in [2]. Using a computer, the author has established that in a set of 1443 graphs with relatively small number of vertices there are 120 pairs, 5 triplets and 1 quadruplet of isospectral non-isomorphic graphs.

The PINGs are known also in the set of regular graphs. Exceptional graphs, having 16, 28 and 64 vertices, from Theorems 3.5, 3.6 and 3.16 belong to such PINGs. In Theorems 3.8, 3.9 and 3.10 some classes of isospectral regular graphs are defined too.

If the PING with n vertices is known, the PING with m $(m>n)$ vertices can easily be constructed by adding an arbitrary graph with $m-n$ vertices as a new component to each graph from the pair. Also from the PING with regular graphs of degree greater than 2 we can construct another PING with graphs having more vertices, if we take line-graphs of the graphs from the given PING.

3.2. Characterizations by spectra

As it has been already said, a graph is not, in general, characterized by its spectrum. Nevertheless, the characterizations of graphs by spectra are possible if we introduce some restrictions.

Let $\mathcal{G}=\{G\}$ be a family of finite graphs. We shall consider the following problems:

1° For the given family $\mathcal{G}$ let us determine the graphs from $\mathcal{G}$ which can be characterized by their spectra with respect to graphs from $\mathcal{G}$. The answers to this question lead to the Theorems of the following type (type A): *Let G, $H\in\mathcal{G}$. If G and H are isospectral graphs, then they are isomorphic too.* Such a statement will be denoted by $A(\mathcal{G}, G)$.

2° Determine the subfamilies of $\mathcal{G}$ not containing the PINGs. To this group theorems of the following type (type B) belong: *Let $\mathcal{H}$ be a given family of graphs. Non-isomorphic graphs from $\mathcal{H}$ have different spectra.* The abbreviation for such a statement is $B(\mathcal{H})$.

3° Determine all graphs from $\mathcal{G}$ having a given spectrum. The correspondent theorems are of the following type (type C): *Let $G, H\in\mathcal{G}$ and let $\mathcal{H}\subset\mathcal{G}$. If H and G are isospectral, then $H\in\mathcal{H}$.* We denote such a statement by $C(\mathcal{G}, G, \mathcal{H})$.

In further text $\mathcal{G}$ represents the set of all finite undirected graphs without loops or multiple edges.

3.2.1. As a particular answer to the first question we quote certain graphs, i. e., classes of graphs, which are characterized by their spectra.

3.2.1.1. The spectrum can characterize all those graphs determined by the number of vertices and the number of edges because these numbers are determinable by the spectra (see **4.1**). To this group belong: graphs without edges, graphs with one edge as well as complements of these graphs.

3.2.1.2. Regular graphs of degree 1, 2, $n-3$ and $n-2$ (n the number of vertices) are determined by the spectrum (for degrees 0 and $n-1$ see **3.2.1.1.**). The regularity of the graph can be established by the spectrum (see Theorem 4.10). Since there exists only one regular graph of degree 1 (for the given even number of vertices), the statement is clear in this case. Since the spectra of regular graphs and their complements are mutually determined (Theorem 2.10), this statement is transferred to the graphs of degree $n-2$).

The case of graphs of degree 2, i. e. $n-3$, is treated in **[30]**, where the following theorem has been proved.

Theorem 3.1. *The connected regular undirected graph G without loops or multiple edges with n vertices of degree $n-3$ is fully determined by its spectrum.*

The author, in fact, proves the accuracy of the statement $A(\mathcal{R}, G)$, where $\mathcal{R}$ is the set of regular graphs (possible with multiple edges). We shall prove the similar theorem formulated for graphs of degree 2 with a new proof of the part which is equivalent to the statement of Theorem 3.1.

Theorem 3.2. *If G is a regular graph of degree 2, then $A(\mathcal{G}, G)$.*

Proof. Let H $(H\in\mathcal{G})$ be a graph isospectral with G. According to theorem 4.10, we can establish that H is regular of degree 2, because this holds for G. Thus, both G and H have, as components cycles. It is necessary to prove still that G and H contain the same cycles. According to Theorem 4.11 the length and the number of the shortest cycles of a regular graph is determinable by the spectrum, and these numbers will naturally be the same for G and H. Then let us delete from both graphs all the shortest cycles and also from the spectra the eigenvalues corresponding to these cycles. The described procedure is iterated until all cycles of graphs G and H are recorded.

This completes the proof of the Theorem.

3.2.1.3. From **[30]** we note the following result.

Theorem 3.3. *The graph G $(G\in\mathcal{R})$ has the characteristical polynomial $P_G(\lambda)=$ $=\left(\lambda+\frac{n}{k}-n\right)\left(\lambda+\frac{n}{k}\right)^{k-1}\lambda^{n-k}$ if and only if G has n vertices and can be represented as ∇-product of k graphs, each containing only a constant number of isolated vertices.*

Complement $\overline{G}$ of the graph G from this theorem has k components, each representing a complete graph with $\frac{n}{k}$ vertices. Since G and $\overline{G}$ are regular graphs, it follows from Theorem 3.3, that graphs representing a direct sum of several identical complete graphs are determined by their spectra too. We shall now prove that the direct sum of the arbitrary complete graphs is determined by its spectrum.

Theorem 3.4. *The graph G $(G\in\mathcal{G})$ has the spectrum containing the natural numbers $n_1-1, \ldots, n_k-1$, s numbers equal to 0, and $n_1+\cdots+n_k-1$ numbers equal to -1, if and only if G can be represented as a direct sum of s isolated vertices and k complete graphs having $n_1, \ldots, n_k$ vertices respectively.*

Proof. According to Example 2.8., the complete graph with $n\,(n>1)$ vertices has the spectrum containing the number $n-1$ as well as $n-1$ numbers equal to -1. Therefrom the first part of the theorem follows.

Suppose now that G has the spectrum described in the theorem. Then G, according to Theorem 1.8, cannot contain, as a subgraph, the bicomplete graph

$K_{1,2}$ because the smallest eigenvalue of $K_{1,2}$ is equal to $-\sqrt{2}$ and this eigenvalue of G is equal to -1. Non-appearance of $K_{1,2}$ implies that the adjacency relation of vertices in G is transitive. This relation is, naturally, symmetric and it can be made reflexive too by adding a loop to each vertex of G. Thus, the adjacency relation is an equivalence. Vertices, corresponding to the classes of the equivalence, form complete graphs or isolated vertices. Complete graphs contain respectively $n_1, \ldots, n_k$ vertices and the number of isolated vertices is equal to s.

This completes the proof of the Theorem.

The transition to the complement of the graph from this Theorem is not always successful as it has been shown in **3.1.** for bicomplete graphs.

3.2.1.4. A group of American and other mathematicians has investigated the question to what extent regular connected graphs are determined by their distinct eigenvalues. In **[42]** A. J. Hoffman has noticed that the investigations from **[87]** lead to the above statement if G is the line graph of the regular bicomplete graph as well as that the results from **[8]**, **[9]**, **[11]**, **[36]**, **[37]**, **[86]**, imply this statement for the case when G is the line graph of the complete graph. According to Theorem 4.10, these results can take the following form.

Theorem 3.5. *Let G be the line graph of the bicomplete graph $K_{n,n}$ $(n \neq 4)$. Then $A(\mathcal{G}, G)$ holds.*

Theorem 3.6. *Let G be the line graph of a complete graph with n $(n \neq 8)$ vertices. Then $A(\mathcal{G}, G)$ holds.*

We shall now prove a generalization of Theorem 3.5. $L(G)$ denotes the line graph of the graph G.

Theorem 3.7. *Let $G = L(K_{n_1,n_2})$, where $n_1 + n_2 \geqq 19$ and where*

$$\{n_1, n_2\} \neq \{2s^2+s, 2s^2-s\},$$

s being a natural number. Then $A(\mathcal{G}, G)$ holds[1].

Proof. The edges of K_{n_1,n_2} can be represented by pairs (i, j) $(i=1, \ldots, n_1, j=1, \ldots, n_2)$, which are the vertices of $L(K_{n_1,n_2})$. Two vertices (i, j) and (k, l) are adjacent in $L(K_{n_1,n_2})$ if and only if either $i=k$, $j \neq l$ or $i \neq k$, $j=l$. According to the definition of the sum of graphs (see **5.1**), $L(K_{n_1,n_2})$ can be represented as the sum of two complete graphs having n_1, i. e., n_2 vertices. According to Theorem 5.3 and on the basis of the known spectra of complete graphs, we see that the spectrum of $L(K_{n_1,n_2})$ contains in the case $n_1, n_2 > 1$ the following numbers: $\lambda = n_1 + n_2 - 2$, $\lambda_2 = n_1 - 2$, $\lambda_3 = n_2 - 2$ and $\lambda_4 = -2$ with multiplicities $p_1 = 1$, $p_2 = n_2 - 1$, $p_3 = n_1 - 1$ and $p_4 = n_1 n_2 - n_1 - n_2 + 1$ respectively. (We omit the case when one of the numbers n_1, n_2 is equal to 1, because then $L(K_{n_1,n_2})$ is a complete graph, determined by its spectrum).

Let graph H have the described spectrum. According to Theorem 4.10, H is a regular connected graph. The degree of the graph is not less than 17. In virtue of Theorem 3.14, we can take $H = L(F)$, where F is a graph without isolated vertices.

---

[1]) This theorem was proved indepedently of **[26]**.

According to a lemma from [72], a regular graph can be the line graph for a regular or for a semi-regular bipartite graph. (The semi regular bipartite graph has two groups of vertices, where none pair of vertices from the same group is adjacent and the vertex degrees in each group are mutually equal).

Suppose that F is regular. Then n_1+n_2 must be an even number and the degree of F is $r=\frac{n_1+n_2}{2}$. The number of edges in F is equal to the number of vertices in H, i. e. $m=n_1 n_2$. The number of vertices is then $n=\frac{2m}{r}=\frac{4n_1 n_2}{n_1+n_2}$. Thus, $4n_1 n_2$ must be divisible by n_1+n_2. Using Theorem 3.12 we see that F contains in the spectrum the numbers: $\frac{n_1+n_2}{2}$, $\frac{n_1-n_2}{2}$, $\frac{n_2-n_1}{2}$, $-\frac{n_1+n_2}{2}$ with multiplicities 1, n_2-1, n_1-1 and $n_1 n_2-n_1-n_2+1-\left(n_1 n_2-\frac{4n_1 n_2}{n_1+n_2}\right)=1-\frac{(n_1-n_2)^2}{n_1+n_2}$. This implies $n_1=n_2$, since the case $(n_1-n_2)^2=n_1+n_2$ leads to $\{n_1, n_2\}=\{2s^2+s, 2s^2-s\}$ what is in contradiction with the assumption of the theorem. However, H has then the spectrum of $L(K_{n_1,n_1})$ and, according to Theorem 3.5, we have $H=L(K_{n_1,n_1})$.

Suppose now that F is semi-regular bipartite. Let F have n_2' mutually non-adjacent vertices of degree r_2' and n_2' mutually non-adjacent vertices of degree r_2', where $n_1'>n_2'$. Then we have

$$n_1' r_1' = n_2' r_2', \qquad n_1' r_1' = n_1 n_2, \qquad r_1' + r_2' = n_1 + n_2. \tag{3.1}$$

On the basis of Theorem 3.13, the spectrum of $L(F)$ must contain the number $r_1'-2$. By comparison with the spectrum of G we have the following possibilities: 1° $r_1'=n_1$; 2° $r_1'=n_2$ and 3° $r_1'=n_1+n_2$. According to (3.1), the alternative 3° leads to an absurd situation and 1° and 2° yield the same solution $F=K_{n_1,n_2}$.

This completes the proof of Theorem 3.7.

3.2.2. A few statements of the type B are known. In [90] J. TURNER proves the statement $B(\mathcal{H})$, where $\mathcal{H}$ is the set of graphs with a prime number of vertices and with a certain symmetry of vertices. Other known statements of the type B are reduced to one which can be formulated on the basis of up to now exposed results.

3.2.3. We quote some known statements of the type C.

On the basis of the results from [42] and Theorem 4.10, the following theorem holds.

Theorem 3.8. *Let G be the line graph of the graph of a projective plane of order n and $\mathcal{H}$ the set of line graphs of graphs of all projective planes of order n. Then $C(\mathcal{G}, G, \mathcal{H})$ holds.*

According to [41] we have the following theorem.

Theorem 3.9. *Let G be the line graph of a finite afine plane of order n and $\mathcal{H}$ the set of line graphs of graphs of all finite afine plane of order n. Then $C(\mathcal{G}, G, \mathcal{H})$ holds.*

Starting from the results of [43] we obtain the following theorem.

Theorem 3.10. *Let G be the line graph of a symmetric balanced incomplete block design with parameters v, k, λ $((v, k, \lambda) \neq (4, 3, 2))$ and $\mathcal{H}$ the set of line graphs of all block designs with the given parameters. Then $C(\mathcal{G}, G, \mathcal{H})$ holds.*

Various cases of characterization can be found also in [25].

3.3. Characterizations by spectral properties. Line graphs.

The following problems belong to this group: *An information on the spectrum of the graph is known. Determine all graphs having the given spectral property.*

Sometimes poor information on the spectrum determines fully the graph. For example if G is a graph with n vertices whose index is equal to $n-1$, then G is a complete graph. In [39] the examples of characterization of graphs by so-called the polynomial of the graph are given. Nevertheless, in majority of cases the given spectral property determines a class of graphs.

The following problem has been considered in several papers: *Determine the class of graphs for which the smallest eigenvalue q from the spectrum satisfies the inequality $q \geqq -2$.* The expository paper [44] is related mainly to this problem. The following theorem is given.

Theorem 3.11. *For the smallest eigenvalue q from the spectrum of the line graph $L(G)$ of the arbitrary graph G the inequality $q \geqq -2$ holds. If G has more edges than vertices, then $q = -2$.*

The characteristic polynomial of $L(G)$ can be expressed by the characteristic polynomial of G if G is a regular graph. The following theorem is proved in [80].

Theorem 3.12. *If G is a regular graph of degree r, with n vertices and $m\left(=\frac{1}{2}nr\right)$ edges, the following relation holds:*

$$P_{L(G)}(\lambda) = (\lambda+2)^{m-n} P_G(\lambda - r + 2). \tag{3.2}$$

In [96] and [100] the analogous relation for characteristic polynomials of matrices $D+A$ and $D-A$ (D the matrix of vertex degrees, A the adjacency matrix) are given.

We shall show that for some more graphs, except for the regular graphs, a relation between $P_G(\lambda)$ and $P_{L(G)}(\lambda)$ can be determined.

Theorem 3.13. *Let G be a semi-regular bipartite graph with n_1 mutually non-adjacent vertices of degree r_1 and n_2 mutually non-adjacent vertices of degree r_2, where $n_1 \geqq n_2$. Then the relation*

$$P_{L(G)}(\lambda) = (\lambda+2)^{\beta} \sqrt{\left(-\frac{\alpha_1}{\alpha_2}\right)^{n_1-n_2} P_G(\sqrt{\alpha_1 \alpha_2})\, P_G(-\sqrt{\alpha_1 \alpha_2})}, \tag{3.3}$$

holds, where $\alpha_i = \lambda - r_i + 2$ $(i = 1,2)$ and $\beta = n_1 r_1 - n_1 - n_2$.

Proof. It is known that for the graph G with n vertices and m edges the relation

$$(3.4)\qquad P_{L(G)}(\lambda-2)=\lambda^{m-n}\det(\lambda I-A-D)$$

holds. For the graph from the theorem we have

$$\det(\lambda I-A-D)=\det\begin{Vmatrix}(\lambda-r_1)I_{n_1} & -K' \\ -K & (\lambda-r_2)I_{n_2}\end{Vmatrix},$$

where K is a $n_2\times n_1$ matrix with entries from the set $\{0, 1\}$. It is known that

$$\det\begin{Vmatrix}M & N \\ P & Q\end{Vmatrix}=\det M\det(Q-PM^{-1}N),$$

for a non-singular square matrix M ([**32**], vol. I, p. 46). Therefore we have

$$(3.5)\qquad \det(\lambda I-A-D)=(\lambda-r_1)^{n_1}\det\left((\lambda-r_2)I_{n_2}-K\frac{I_{n_1}}{\lambda-r_1}K'\right)$$

$$=(\lambda-r_1)^{n_1-n_2}\det\big((\lambda-r_1)\ (\lambda-r_2)I_{n_2}-KK'\big)$$

$$=(\lambda-r_1)^{n_1-n_2}P_{KK'}\big((\lambda-r_1)\ (\lambda-r_2)\big).$$

The characteristic polynomial $P_{KK'}(\lambda)$ of the matrix KK' can be expressed by the characteristic polynomial of the adjacency matrix A. At first we have

$$A=\begin{Vmatrix}0 & K' \\ K & 0\end{Vmatrix},\qquad A^2=\begin{Vmatrix}K'K & 0 \\ 0 & KK'\end{Vmatrix}.$$

The relation $P_{K'K}(\lambda)=\lambda^{n_1-n_2}P_{KK'}(\lambda)$ is known in the matrix theory (see, for example, [**59**], p.24). Since $P_{A^2}(\lambda)=P_{K'K}(\lambda)P_{KK'}(\lambda)$ and since the eigenvalues of A^2 are squares of eigenvalues of A, i.e., $P_{A^2}(\lambda^2)=(-1)^{n_1+n_2}P_A(\lambda)P_A(-\lambda)$, we have

$$(3.6)\qquad P_{KK'}(\lambda)=\sqrt{\frac{P_{A^2}(\lambda)}{\lambda^{n_1-n_2}}}=\sqrt{(-1)^{n_1+n_2}\lambda^{n_2-n_1}P_A(\sqrt{\lambda})P_A(-\sqrt{\lambda})}.$$

Combining expressions (3.4) — (3.6) we get (3.3).

This completes the proof of the theorem.

In [**44**] and [**45**] the following theorem from, apparently, up to now unpublished paper [**47**] is mentioned:

Theorem 3.14. *Let G be a regular connected graph of degree not less than* 17 *and with $q\geqq -2$. Then G is either the complement of the regular graph of degree* 1 *or a line graph. Number* 17 *is the best possible.*

An analogous theorem without the assumption of regularity is given in [**72**]. The degree of vertex u is denoted by $d(u)$. Let $d(G)$ denote the smallest vertex degree from G and $\triangle(u, v)$ the number of vertices adjacent to both of vertices u and v.

Theorem 3.15. *If for the graph G the following holds:* a) $d(G)>43$, b) $q=-2$, c) *for non-adjacent vertices u_1 and u_2 it is $\triangle(u_1, u_2)<d(u_i)-2$ $(i=1, 2)$, then there exists a graph H such that $G=L(H)$. Inversely, if $G=L(H)$ with $d(H)>3$, then G satisfies conditions* b) *and* c).

3.4. Characterizations of the cubic lattice graph

This section is contained in **[19]**. We give only the survey of results.

A cubic lattice graph with characteristic $n(n>1)$ is a graph whose vertices are all the n^3 ordered triplets of n symbols, with two triplets adjacent if and only if they differ in exactly one coordinate. Let $d(x, y)$ denote the distance between two vertices x and y, $\triangle(x, y)$ the number of vertices adjacent to both x and y, and $n_2(x)$ the number of vertices at the distance 2 from x. We list some properties of the cubic lattice graph G: (P_1) The number of vertices is n^3; (P_2) G is connected and regular; (P_3) $n_2(x)=3(n-1)^2$ for all x in G; (P_3''') $\triangle(x, y)>1$ for all x, y such that $d(x, y)=2$; (P_4) The distinct eigenvalues of the adjacency matrix of G are $3n-3$, $2n-3$, $n-3$, -3; (P) The adjacency matrix of G has eigenvalues $\lambda_f=3n-3-fn$ $(f=0, 1, 2, 3)$ with multiplicities $p_f=\binom{3}{f}(n-1)^f$.

Theorem 3.16. *Graph G is the cubic lattice graph with characteristic* $n(n\neq 4)$ *if and only if it has properties* (P_1), (P_2), (P_3) *and* (P_4).

Theorem 3.17. *Graph G is the cubic lattice graph with the characteristic* $n(n\neq 4)$ *if and only if it has properties* (P_1), (P_2), (P_3''') *and* (P_4).

Theorem 3.18. *Graph G is the cubic lattice graph with characteristic* $n(n\neq 4)$ *if and only if it has properties* (P) *and* (P_3''').

It can be proved that properties (P_1), (P_2) and (P_4) are equivalent with (P).

Theorem 3.16 was proved in **[56]** for $n>7$. A similar result for the tetrahedral graph can be found in **[5]**. A method for determination of eigenvalues in similar graphs is described in **[66]**.

4. DETERMINATION OF THE STRUCTURAL DETAILS OF A GRAPH BY THE USE OF THE SPECTRUM OF THE GRAPH

In this Chapter we expose theorems which enable the determination of various structural properties of a graph provided the spectrum of the graph is known. In some theorems we use the eigenvectors of the adjacency matrix too. Some structural properties are not, naturally, uniquely determined by the spectrum, but we can often in these cases also precise, on the basis of the spectrum, the field of variation of these properties. The contents of this Chapter, together with that of the preceding one, represent the basis for the investigation of graphs by use of spectra. Section **4.5.** contains concluding remarks on the spectral method and on the possibilities of its application.

In all the theorems we assume that either the spectrum or the eigenvectors of the adjacency matrix of a graph, or both, are given and that a class of graphs to which the given graph belongs is precised. If the spectrum of the graph is given, we assume that its characteristic polynomial is known too, and vice versa. The algebraic and numerical problems, which appear here, are assumed to be solved. Note that in some cases the class of graphs to which the graph with the given spectrum belongs can be determined by the spectrums.

4.1. Analysis of an arbitrary graph structure

Let the characteristic polynomial of an arbitrary graph be given:

$$P_G(\lambda)=\det(\lambda I-A)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n. \tag{4.1}$$

Before we restrict ourselves to undirected graphs without loops or multiple edges, it is necessary to list the following possibilities for graph analysis:

1° The number of vertices is equal to the degree n of the characteristic polynomial. The number of loops is equal to the trace of the adjacency matrix, i.e. to the quantity $-a_1$.

2° If we know that the graph G belongs to the class of graphs in which the number of loops is constant for every vertex, then the characteristic polynomial of the graph H, obtained from G by deleting all loops, can be determined. Let every vertex in G have h loops. Then $h=-\frac{a_1}{n}$ and $P_H(\lambda)=P_G(\lambda+h)$.

3° If G belongs to the class of graphs witout loops, then G is a digraph if and only if $a_2=0$. This fact can be easily seen by considering all principal minors of the second order of the adjacency matrix.

4° H. - J. Finck shows in **[30]**, that a graph without multiple edges can be recognized in the class of regular graphs without loops if the spectrum is known. The formula $z=-\frac{1}{4}(nr+2a_2)$ was proved, where z denotes the number of cycles of length 2, and r is the index of the graph. Graph is without multiple edges if and only if $z=0$.

We consider further only undirected graphs without loops or multiple edges. At present, $a_1=0$ always holds.

L. Collatz and U. Sinogowitz give in **[10]** the following statements: 1° The number of edges of the graph is equal to $-a_2$; 2° The number of triangles in the graph is equal to $-\frac{1}{2}a_3$.

H. Sachs proved in **[78]** the following theorem:

Theorem 4.1. *The length k of the shortest cycle of an odd length in the graph G is equal to the index of the first of the coefficients $a_3, a_5, a_7, \ldots$ which is different from zero; the number of all shortest cycles of odd length is equal to* $-\frac{1}{2}a_k$.

All the results listed up to now in this Section are contained in Theorems 1.1 and 1.2. Other specializations of these theorems do not lead to statements which can be used in general case.

According to Theorem 2.1, it is possible to determine by spectrum of the graph the number of closed walks in the graph. This number is equal to $tr\,A^k=\sum_{i=1}^{n}\lambda_i^k$ (k natural number).

An example of interest, of the connection between the structure and the spectrum of a graph is given by the following theorem.

Theorem 4.2. *A graph, containing at least one edge, is bipartite if and only if its spectrum, taken as a set of points on the number axis, is symmetric with respect to the zero point.*

This theorem is, in various versions and for various classes of graphs, known in the literature ([10], [39], [60], [78], [79]). In [12] a new proof of it is given. However, by the method of [12] it is possible to prove also the following theorem.

Theorem 4.3. *The connected graph, containing at least one edge, with the index r is bipartite if and only if its spectrum contains the number $-r$.*

Proof. According to Theorem 4.2, it is sufficient to prove that the bipartity of the graph is a consequence of the presence of the number $-r$ in the spectrum. The eigenvalue r is simple according to Theorem 1.4. The number $-r$ is also a simple eigenvalue, because the opposite case would be in collision with Theorem 1.4. Hence, the adjacency matrix A has exactly two eigenvalues of the maximal modulus r. Thus, according to Theorem 1.4. there exists a permutation matrix P such that

$$P^{-1}AP=\left\|\begin{matrix}0 & A_{12}\\ A_{21} & 0\end{matrix}\right\|,$$

where the square zero matrices are on the main diagonal.

This completes the proof of the theorem.

This theorem is known in the literature only for regular graphs ([39], [79]).

The spectrum provides further information about the graph in the following way. By CAYLEY-HAMILTON's theorem from the matrix theory we get in virtue of (4.1), the following relations:

$$A^{n+k}+a_1A^{n+k-1}+\cdots+a_nA^k=0 \quad (k=0,\,1,\,\ldots). \tag{4.2}$$

Since the adjacency matrix is a real symmetric matrix, the roots of its minimal polynomial are also roots of the characteristic polynomials (and vice versa) and are mutually distinct. So we have the relations

$$b_0A^{m+k}+b_1A^{m+k-1}+\cdots+b_mA^k=0 \quad (k=0,\,1\,\ldots), \tag{4.3}$$

where $b_0, b_1, \ldots, b_m$ are coefficients of the minimal polynomial and m is the number of mutually distinct eigenvalues in the spectrum of the graph. By means of Theorem 2.1 several information on graph can be obtained from (4.2) and (4.3) (see, for example, [19], where a similar relation was used).

The spectrum of the graph allows one to obtain some inequalities for several characteristics of the graph. We list some possibilities.

L. COLLATZ and U. SINOGOWITZ give in [10] the following theorem.

Theorem 4.4. *Let $d_{\min}$, $\bar{d}$ and $d_{\max}$ be the minimal, mean and maximal value for the degrees of vertices in the connected graph G whose index is equal to r. Then $d_{\min}\leqq\bar{d}\leqq r\leqq d_{\max}$. Equality holds in $\bar{d}\leqq r$ if and only if G is a regular graph.*

This theorem represents, in fact, a reformulation of a theorem from the matrix theory (see, for example, [32], vol II, p. 63). It is of interest to note that applications of several other inequalities for eigenvalues of matrices do not provide better results.

Theorem 4.5. *If the graph G has m mutually different eigenvalues, then G is either an unconnected graph or the inequality* $D \leqq m-1$ *holds for its diameter D.*

We omit the proof since there is an analogous theorem in the matrix theory (see, for example, [59], p. 125).

H. S. WILF proves in [94] the following theorem.

Theorem 4.6. *If* γ *is the chromatic number and r the index of a connected graph, then* $\gamma \leqq 1+r$*, where equality holds if and only if the graph is complete or if it represents a cycle of odd length.*

Note that $\gamma \leqq 1+r$ holds also for undirected graphs but then the conditions for equality are formally more complex. J. MITCHEM [62] and D.R. LICK [57] generalized this result in a certain way. See also [49], [50].

According to Theorems 4.6. and 4.2., we have the following theorem.

Theorem 4.7. *The chromatic number* γ *can be determined by the graph spestrum if the graph has the index r with* $r<3$*. If the graph is connected, the same statement holds also for* $r=3$.

On the basis of the spectrum the inequalities for the number of internal stability $\alpha(G)$ and the number $k(G)$ of vertices of the maximal complete subgraph of the graph G can be obtained.

Theorem 4.8. *The number of internal stability* $\alpha(G)$ *of the graph G satisfies the inequality* $\alpha(G) \leqq p_o + \min(p_-, p_+)$*, where* p_-, p_o, p_+ *denote the numbers of eigenvalues, smaller, equal and greater than zero in the spectrum of G.*

Theorem 4.9. *Let* $p_{\lambda<-1}, p_{-1}$ *and* $p_{\lambda>-1}$ *denote the numbers of eigenvalues smaller, equal and greater than —1 in the spectrum of the graph G.* λ *represents the smallest eigenvalue greater than —1. Let us further have* $p = p_{\lambda<-1} + p_{-1} + 1$ *and* $s = \min(p, p_{\lambda>-1} + p_{-1}, r+1)$*, where r is the index of G. If* $k(G)$ *denotes the number of vertices of the maximal complete subgraph of G, then the inequality*

$$k(G) \leqq \begin{cases} s, & s<p \\ s-\alpha_\lambda, & s=p, \end{cases}$$

holds, where $\alpha_\lambda = 0$ *for* $\lambda \leqq p-1$ *and* $\alpha_\lambda = 1$ *for* $\lambda > p-1$.

The proofs of these two theorems use Theorem 1.9 [22]. In [22] an inequality for the chromatic number of a graph is obtained too.

The connection between the automorphism group and the spectrum of a graph has been studied in [10], [28], [64], [65], [68], [69], [70].

4.2. Analysis of the regular graph structure

We begin with the question in which way a regular graph can be recognized by its spectrum.

Theorem 4.10. *The spectrum of a graph contains information about its possible regularity. Let* $\{\lambda_1 = r, \lambda_2, \ldots, \lambda_n\}$ *be the spectrum of the graph G, where r denotes the index of G. G is regular if and only if*

$$\frac{1}{n}\sum_{i=1}^{n} \lambda_i^2 = r. \tag{4.4.}$$

If (4.4) *holds and if r has multiplicity p, the graph has p components.*

This theorem is implicitely contained in **[10]**. See also **[19]**. Theorem 4.10 is quite important in several places in this paper. Note that Theorem 4.10 enables the determination of the cyclomatic number and the establishment of the existence of the Eulerian closed walks in a graph provided we have primarily established, by the mentioned theorem, the graph to be regular.

In the following theorems we require that graph is: 1° regular, or 2° regular and connected. These conditions can be replaced by the following: 1° The spectrum of the graph satisfies relation (4.4), 2° The spectrum of the graph satisfies relation (4.4) and r is a simple eigenvalue. Thus, in the following theorems the non-spectral information about the graph structure appears only apparently.

In **[78]** H. Sachs has arrived at the following result.

Theorem 4.11. *If G is a regular graph, then by* $P_G(\lambda)$ *the length t of the shortest cycle in G as well as the number of cycles of length* h $(h \leqq 2t-1)$ *can be determined.*

Further, we have the following important result of A.J. Hoffman **[39]**:

Theorem 4.12. *For the graph G with adjacency matrix A there exists a polynomia* $P(x)$*, such that* $P(A) = J$ *if and only if G is regular and connected. In this case we have*

$$P(x) = \frac{n(x-\lambda_2)\cdots(x-\lambda_m)}{(r-\lambda_2)\cdots(r-\lambda_m)} \tag{4.5}$$

where n is the number of vertices, r is the index and $\lambda_1 = r, \lambda_2, \ldots, \lambda_m$ *are all mutually distinct eigenvalues from the spectrum of G.*

This result was extended in **[40]** by A.J. Hoffman and M.H. McAndrew to directed graphs.

This theorem provides great possibilities for investigation of the structure of graphs by means of spectra. The examples of such an utilization of Theorem 4.12 are found in **[19]** as well as in the papers quoted there.

Further, it is know that the characteristic polynomial of a regular graph contains information about the number of spanning trees in the graph. This fact is a direct consequence of the well known theorem of Kirchhoff—Trent (see, for example, **[3]**, *p.* 152). It is established in **[52]**, where it is noticed that the number of spanning trees of a regular graph G of degree r with n vertices is given by the formula

$$D(G) = \frac{1}{n} P_G'(r). \tag{4.6}$$

A more general result is given in **[100]**. On the basis of (1.2) we can formulate Theorem 1 from **[100]**, for the regular graphs in the following way.

Theorem 4.13. *Let G be a regular graph of dergee r with X being its set of vertices; let $Y \subset X$ and let G_Y represent the graph obtained by mutually identifying all vertices from Y and by deleting all the loops then occurring. Then*

$$\sum_{\substack{Y \subset X \\ |Y|=k}} D(G_Y) = \frac{1}{k!} P_G^{(k)}(r), \tag{4.7}$$

where $D(H)$ denotes the number of spanning trees in H and $|Y|$ denotes the number of elements of the set Y.

For $k=1$ in the sum on the left side every tree of G is taken exastly n times and therefore (4.7) reduces to (4.6).

EXAMPLE 4.1. The characteristic polynomial of the complete graph G_1 with n vertices and the regular graph G_2 of degree $n-2$ with n vertices were determined in example 2.9 and according to (4.6) we have $D(G_1) = n^{n-2}$ (the well known CAYLEY's formula) and $D(G_2) = (n-2)^{\frac{n}{2}} n^{\frac{n}{2}-2}$.

Further examples of determining the number of trees are given in **[21]**.

In **[29]** H. - J. FINCK and G. GROHMANN prove the following theorem by use of which it can be established whether or not a graph is ∇-prime.

Theorem 4.14. *Let G be a regular connected graph of degree r with n vertices. G can be represented as ∇-product of $p+1$ ∇-prime graphs if and only if $r-n$ is a p-fold eigenvalue in the spectrum of G.*

EXAMPLE 4.2. (H.—J. FINCK and G. GROHMANN) If the number $r-n$ exists in the spectrum of a regular graph, then $r-n \geqq -r$, i. e. $n \leqq 2r$. Thus almost all regular graphs of a given degree r are ∇-prime.

EXAMPLE 4.3. Determine all regular graphs G being not ∇-prime and belonging to the set of line-graphs.

According to Theorem 3.11 we have $q \geqq -2$ for the smallest eigenvalue q from the spectrum of G. Thus, G is not ∇-prime if either $r-n=-1$ or $r-n=-2$, holds. In the first case we have $r=n-1$, i. e., G is the complete graph with $n>1$. In the second case n is even and according to **[44]** G does not belong to the set of line graphs for $n>6$. Hence, G can be, except for complete graphs, a cycle of length 4 and a regular graph of degree 4 with 6 vertices.

In **[30]** H. - J. FINCK investigates the connection between the chromatic number and the spectrum for regular graphs which are developing into product of two or more ∇ - prime factors. The main results of this paper are expressed by the following theorem.

Theorem 4.15. *Let G be a regular connected graph of degree r with n vertices $(n>r+1)$. Let p_λ be the algebraic multiplicity of the eigenvalue λ from the spectrum of G. Then the following relations for the chromatic number γ of G hold:*

$$\gamma \geqq p_{r-n} + 1; \tag{4.8}$$

$$\gamma \geqq 2(p_{r-n}+1) - \left[\frac{p_0}{n-r-1}\right]; \tag{4.9}$$

$$\gamma \geqq 3(p_{r-n}+1) - 2\left[\frac{p_0}{n-r-1}\right] - \tag{4.10}$$

$$-\varepsilon_{n-r} \min\left(\left[\frac{2p_{-1}}{n-r+1}\right], \left[\frac{2p_1}{n-r-1}\right]\right) - \sum_{v=r-n+2}^{-2} p_v \left(\varepsilon_s = \frac{1}{2}\left(1+(-1)^{s-1}\right)\right);$$

(4.11) $$\gamma \leqq (p_{r-n}+1)\ (r+1)-np_{r-n};$$

(4.12) $$\gamma \leqq (p_{r-n}+1)\,r-np_{r-n}+\left[\frac{p_0}{n-r-1}\right]+p_2+\left[\frac{p_{-1}}{3}\right].$$

The inequality (4.10) is improved in **[54]**.

In **[30]** a formula, giving the unique connection between the chromatic number and the spectrum of a regular graph of degree $n-3$ with n vertices, is proved too.

In **[79]** and **[80]** H. SACHS describes the use of the characteristic polynomial of a graph for a special colouring of vertices of the graph. This colouring is related to the problem of factorization of a graph.

In **[33]**, **[58]** and **[83]** — **[85]** (J. J. SEIDEL and others) the properties of strongly regular graphs (introduced in **[4]**), are connected with the spectrum of $(-1, 1, 0)$ — adjacency matrix of the graph. The results can be reformulated in terms of the spectrum of the usual adjacency matrix.

4.3. Graph analysis by means of the spectrum and of the eigenvectors

If besides the spectrum of the graph, the eigenvectors of the adjacency matrix are known, the number of statements which can be made about the graph structure increases naturally. Sometimes valuable information about the graph can be attained by knowing eigenvectors only. Such a result is given by the following theorem.

Theorem 4.16. *A graph is regular if and only if its adjacency matrix has an eigenvector whose all coordinates are equal to* 1.

This theorem is known in the matrix theory (see, for example, **[59]**, p. 133).

In **[3]**, p. 131, the following result of T. H. WEI **[92]** is noticed:

Let N_{ki} be the number of walks of length k starting from the vertex x_i $(i=1, \ldots, n)$ of a connected graph. Let $s_{ki}=\dfrac{N_{ki}}{\sum_{i=1}^{n} N_{ki}}$. When $k\to+\infty$, the vector $(s_{k1}, \ldots, s_{kn})$ tends towards the eigenvector of the graph index.

The connectedness of the undirected graph can be investigated by Theorem 1.10. Combining Theorems 1.4 and 1.10 we have the following theorem.

Theorem 4.17. *An undirected graph is connected if and only if its index is a simple eigenvalue with a positive eigenvector.*

Theorem 1.11 can also be translated in the graph theory language.

Theorem 4.18. *If the index of an undirected graph G has the multiplicity p and if a positive eigenvector corresponds to it, then G has p components.*

These theorems are used in 5.3.

In papers [36], [37], [43], [44], [45], [47], [72] of A. J. HOFFMAN and D. E. RAY-CHAUDHURI the following interesting method for obtaining information about the graph by means of the spectrum is used.

Let G be a regular graph having the smallest eigenvalue $q=-2$, and let H be the graph for which $q=-2$ also holds and where corresponding eigenvector x has coordinates whose sum is not equal to zero. Then, according to Theorem 1.8, H cannot be a subgraph of G because in this case the vector x would be orthogonal to the projection of the eigenvector of the index of G on the subspace corresponding to H. However, the eigenvector of the index of G has all coordinates equal to 1 and the orthogonality condition leads to the fact that the sum of coordinates of x is equal to zero, which is contradictory to the assumption.

On the basis of this and on the basis of a similar principle ([43]), an extensive list of impossible subgraphs for regular graphs with $q=-2$ is established in the mentioned papers. It would be of interest to form similar lists for other values of q. Some impossible subgraphs under other conditions are given in [51].

4.4. Establishment of non-existence of some properties for a given graph

Let S be a spectral and Q a structural property of a graph and let S be a necessary condition for Q. Then the non-existence of S implies non-existence of Q. The statements of this type supply some information about the graph structure. In this Section we quote some theorems of this kind.

To this group belong, in fact, all statements related to the impossible subgraphs. They can be expressed by the following theorem.

Theorem 4.19. *If $q=-2$ for the regular graph G, then G contains, as a subgraph, none of graphs listed in previously cited papers.*

A. J. Hoffman ([44], [45] and [48]) has posed and partially solved the following problem. Let G be a given graph, let H be any graph containing G as a subgraph and let H_R be an arbitrary regular graph containing G as a subgraph. $d(F)$ denotes the smallest vertex degree for the graph F and $\lambda(F)$ the smallest eigenvalue from the spectrum of F. Determine:

$$\lambda_R(G)=\sup_{H_R}\lambda(H_R),\quad \mu(G)=\lim_{d\to\infty}\sup_{\substack{H\\ d(H)>d}}\lambda(H),\quad \mu_R(G)=\lim_{d\to\infty}\sup_{d(H_R)>d}\lambda(H_R).$$

Familiarity with these quantities (existing for any graph G) leads, for example, to the statements of the following type:

Theorem 4.20. *If G is a regular graph for which $q>\lambda_R(H)$ holds, then G does not contain the graph H as a subgraph.*

Contrary to theorems from **4.1** and **4.2.** theorems from this Section cannot always be applied. However, in special cases they can be very useful. We quote some possibilities more:

1° As it was already said, the information about connectedness of a graph is not in general case contained in the spectrum of the graph. However, if, for example, the index of the graph is not a simple eigenvalue, then the graph is obviously unconnected.

2° Corollary 2 of Theorem 2.9 can be represented in the following way: The graph G, having eigenvalues λ_i and λ_j ($\lambda_i+\lambda_j=-1$) whose multiplicities differ for more than 1, is not self-complementary.

3° The following statement is in [96] (E. V. Vahovsky) formulated for the matrix $A+D$: Let Y be a subset of the set of vertices of the graph G and let the vertices x_1 and x_2, not belonging to Y, be adjacent to each of vertices from Y, and solely to them. Then the spectrum of G contains the number 0 if x_1 and x_2 are not adjacent and the number -1 if x_1 and x_2 are adjacent.

4.5. On possibilities of spectral method applications

On the basis of the exposed matter we consider that the main variant of the spectral method is the one using the spectrum of the adjacency matrix of the graph, though other matrices have the advantage in some problems.

Estimation of the importance of the spectral method will depend on the number and importance of results which may by achieved by it. It seems that only initial steps into the application of this method are made. An interesting example of the application of the spectral method represents the investigation of the existence of certain class of graphs (A. J. HOFFMAN and R. R. SINGLETON [38], R. SINGLETON [88], W. BROWN [6] and H. SACHS [77]). The results obtained by the spectral method are found, for example, in papers [14], [15], [17], [20], [24], [36], [37], [55], [80], [94], [98], [99] and [100]. In this paper the results from Chapter 2 (Theorem 2.5) from Chapter 4 (examples 4.1 and 4.3) and from Chapter 5 (Theorems 5.5—5.11) belong to this group.

Note that several known results in graph theory can be deduced by the spectral method.

There are two aspects of application of the spectral method.

In the first case we have the following scheme. By the group of structural properties A a class of graphs is precised and we search for the structural properties of this class of graphs from the group of structural properties B. By the group A we determine the spectrum or some spectral properties of these graphs. From the spectrum the interesting facts for the group of properties B may be read.

The use of this scheme requires the development of procedures for determining the spectra of graphs. In fact, the spectra are known only for certain narrow classes of graphs. It would be of interest to compile all usable results and statements from the theory of determinant and matrices. Of great importance are several compositions of graphs, as for example, are the product and the sum of graphs, (see **5.1.**) which enable the determination of the spectrum of a compound graph by spectra of some simpler graphs.

The second aspect of the spectral method is related to the use of a computer. If a concrete graph is given (not a class of graphs) and if it is necessary to investigate some of its structural details, it is possible to use a computer under the condition that the number of vertices is not too great. (If the graph has less than, for instance, 10 vertices, majority of tasks can be solved by "hand"). In the literature several computer algorithms for various problems on graphs are described. The use of the spectral methods is probably inefficient for computer solving of only one problem on the graph, because the numerical determination of the spectrum on the computer can also take a long time. However, if it were necessary to solve simultaneously several various problems on a graph, it seems that the use of the spectral method could be taken into account. General testing computer programme for the graphs could be made, which would provide on the basis of the spectrum all information on the graph liable to be obtained from the spectrum. Having in view the memory capacity and the rapidity of recent computers, it seems that the spectral methods would provide satisfying results for the graphs having 10—100 vertices.

5. ON A CLASS ON n-ARY OPERATIONS ON GRAPHS

This chapter describes an example of application of the spectral method, by which some properties of a class of n-ary operations on graphs are examined. The procedure is, basically, the following. We determine a relation between the spectrum of the graph G, obtained as the result of n-ary operation and spectra of graphs $G_1, \ldots, G_n$ on which operation is made. Using theorems, proving the connection between the spectral and the structural properties of graphs, we connect, further, the structural properties of G with structural properties of graph $G_1, \ldots, G_n$. Mainly, connectedness, bipartity and the numbers of walks are investigated.

Operations from this Chapter represent a generalization of some operations described in literature.

This Chapter is related to papers [**13**], [**14**], [**15**], [**16**], and [**20**] and forms an entirety with them.

5.1. Definitions of operations

We shall consider n-ary operations defined on the set of finite, undirected graphs without loops or multiple edges, in which the result is a graph whose set of vertices is equal to the Cartesian product of sets of vertices of graphs on which the operation is made. Known operations of this kind are product, sum and p-sum of graphs ([**3**], p. 23 and p. 53). In [**15**] definitions of these operations are given too, as well as definitions of some of its generalizations. All the definitions are formulated for undirected graphs without loops or multiple edges. However, defined operations can be considered on some other class of graphs, too.

Note that p-sum, with its special cases, appears in several papers under several names. So, for example, product of two graphs is called: product ([**3**], p. 23), Cartesian product ([**67**]), KRONECKER's product ([**93**]), conjuction ([**35**]), cardinal product ([**23**], [**61**]) and so on.The same may be said for the sum of graphs too.

In [**15**] the incomplete p-sum and the incomplete extended p-sum of graphs are defined. Since the first of these operations is a special case of the second, we consider only the latter. The following definition of the incomplete extended p-sum (briefly: NEPS according to Serbo-Croat: *nepotpuna proširena p-suma*) is different but equivalent with that from [**15**]. The equivalence will be shown in the following Section (Theorem 5.1).

Let B be a set of n-tuples $(\beta_1, \ldots, \beta_n)$ of symbols 0 and 1, which do not contain n-tuple $(0, \ldots, 0)$.

Definition 5.1. NEPS *with the basis B of the graphs $G_1, \ldots, G_n$ is the graph, whose set of vertices is equal to the Cartesian product of the sets of vertices of the graphs $G_1, \ldots, G_n$ and in which two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if and only if there is a n-tuple $(\beta_1, \ldots, \beta_n)$ in B such that $x_i = y_i$ holds exactly when $\beta_i = 0$ and x_i is adjacent to y_i in G_i exactly when $\beta_i = 1$.*

We shall quote also the Boolean operations on graphs. We give a definition, according to [**20**]. In [**35**] differently defined Boolean operations on graphs are considered.

Definition 5.2. *Let $G_i = (X_i, U_i)$ $(i = 1, \ldots, n)$ be given graphs, where X_i and U_i denote corresponding sets of vertices and of edges. If $f(p_1, \ldots, p_n)$ is the arbitrary Boolean function $(f: \{0, 1\}^n \to \{0, 1\})$, the Boolean function $G = f(G_1, \ldots, G_n)$ of the graphs $G_1, \ldots, G_n$ is the graph $G = (X, U)$, where $X = X_1 \times \cdots \times X_n$ and where U is defined in the following way. For arbitrary two vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ from G the Boolean variables $p_1, \ldots, p_n$ are defined where, for every i, $p_i = 1$ if and only if x_i and y_i are adjacent in G_i. The vertices $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent in G if and only if, for every i, $x_i \neq y_i$ and $f(p_1, \ldots, p_n) = 1$.*

The set of n-tuples, for which the Boolean function $f(p_1, \ldots, p_n)$ takes the value 1, we denote by F. We utilise also the abbreviation $\beta = (\beta_1, \ldots, \beta_n)$.

Definition 5.3. NEPS *with the basis B of graphs $G_1, \ldots, G_n$ is corresponded to the Boolean function $f(G_1, \ldots, G_n)$ $(f(p_1, \ldots, p_n) \not\equiv 0)$, if $B = F \setminus (0, \ldots, 0)$.*

5.2. Adjacency matrices and spectra

Some data on adjacency matrices of product, sum and p-sum of graphs are given in **[16]**. Except this, note that in **[23]** adjacency matrix for product of graphs is also given. In **[35]** adjacency matrices of a number of binary operations on graphs are listed.

Adjacency matrices for the considered operations on graphs are expressed in terms of the adjacency matrices of graphs, on which operations are made, by the use of KRONECKER's multiplication of matrices. KRONECKER's multiplication of matrices will be denoted by $\otimes$. The properties of KRONECKER's product of matrices, used in further text, are given in **[13]**.

Theorem 5.1. NEPS G *with basis* B *of graphs* $G_1, \ldots, G_n$, *whose adjacency matrices are* $A_1, \ldots, A_n$, *has the following adjacency matrix*

$$\mathcal{A} = \sum_{\beta \in B} A_1^{\beta_1} \otimes \cdots \otimes A_n^{\beta_n}. \tag{5.1}$$

Proof. Let in every of graphs $G_1, \ldots, G_n$ vertices be ordered (numbered). We shall give lexicographic order to the vertices of G (which represent the ordered n-tuples of vertices of graphs $G_1, \ldots, G_n$) and we form adjacency matrix $\mathcal{A}$ according to this ordering.

If x and y are vertices of arbitrary graph with adjacency matrix A, we denote by $(A)_{xy}$ the element of A from the row corresponding to x and column corresponding to y.

In virtue of properties of KRONECKER's product of matrices we then have

$$a \overset{\text{def}}{=} (\mathcal{A})_{(x_1, \ldots, x_n), (y_1, \ldots, y_n),} = \sum_{\beta \in B} (A_1^{\beta_1})_{x_1 y_1} \cdots (A_n^{\beta_n})_{x_n y_n}. \tag{5.2}$$

If $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent vertices, then, according to the definition of NEPS, there is a n-tuple β in B such that $x_i = y_i$ only for $\beta_i = 0$ and such that x_i is adjacent to y_i only for $\beta_i = 1$. Such n-tuple is obviously, unique in B. The corresponding summand in (5.2) is then equal to 1, because each of the factors $(A_i^{\beta_i})_{x_i y_i}$ is equal to 1. Namely, if $\beta_i = 0$, $A_i^{\beta_i}$ is equal to the unit matrix, but then $x_i = y_i$ also holds. For $\beta_i = 1$, x_i and y_i are adjacent vertices and we have $(A_i^{\beta_i})^{x_i y_i} = (A_i)_{x_i y_i} = 1$. All other summands in (5.2) are equal to 0 and we have $a = 1$.

Let now $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be non-adjacent vertices. Then, for every β from B there exists an i such that either $\beta_i = 1$ and x_i is not adjacent to y_i or $\beta_i = 0$ and $x_i \neq y_i$. Therefore, we have $(A_i^{\beta_i})_{x_i y_i} = 0$ and $a = 0$.

This completes the proof of the theorem.

The adjacency matrix of the p-sum is obtained from (5.1) if B contains all n-tuples with p unities. By further specialization, we get the product of graphs for $n = 2$, $p = 2$ and the sum of graphs for $n = 2$, $p = 1$.

The following theorem is proved in **[20]**.

Theorem 5.2. *The Boolean function* $G = f(G_1, \ldots, G_n)$ *of graphs* $G_1, \ldots, G_n$, *having adjacency matrices* $A_1, \ldots, A_n$, *has the adjacency matrix*

$$A = O + \sum_{\beta \in F} A_1^{[\beta_1]} \otimes \cdots \otimes A_n^{[\beta_n]}, \tag{5.3}$$

where O is a zero-matrix of the corresponding order and where the convention $A_i^{[1]}=A_i$, $A_i^{[0]}=\overline{A}_i$ holds for matrices; $\overline{A}_i$ being the adjacency matrix for the complement $\overline{G}_i$ of the graph G_i.

The following two theorems describe the relation between the spectrum of NEPS or the Boolean function and the spectra of graphs on which operations are made. For the Boolean function the relation is obtained only for the case when $G_1, \ldots, G_n$ are regular graphs.

Let graphs $G_1, \ldots, G_n$ have respectively $m_1, \ldots, m_n$ vertices and let $\{\lambda_{ji_j} \mid i_j=1, \ldots, m_j\}$, $j=1, \ldots, n$ be their spectra. Let the eigenvector u_{ji_j} correspond to the eigenvalue λ_{ji_j}.

Theorem 5.3. *NEPS with the basis B of graphs $G_1, \ldots, G_n$, having the spectra $\{\lambda_{ji_j} \mid i_j=1, \ldots, m_j\}$ $(j=1, \ldots, n)$, has the spectrum $\{\Lambda_{i_1,\ldots,i_n} \mid i_j=1, \ldots, m_j;$ $j=1, \ldots, n\}$, where*

$$(5.4) \qquad \Lambda_{i_1,\ldots,i_n}=\sum_{\beta\in B}\lambda_{1i_1}{}^{\beta_1}\cdots\lambda_{ni_n}{}^{\beta_n}.$$

This theorem is a direct consequence of Theorem 5.1 and the Lemma from **[13]**. Some special cases of this theorem were quoted in **[13]** and have been known in the matrix theory. Specially, for the sum of graphs we refer to **[75]**. Note, that the spectrum of the p-sum is equal to the set of all values of the elementary symmetric function of order p of variables $\lambda_{1i_1}, \ldots, \lambda_{ni_n}$.

We proceed to the determination of the spectrum of the Boolean function of graphs. We assume that $\lambda_{j1}=r_j$ $(j=1, \ldots, n)$, where r_j are indices of graphs $G_1, \ldots, G_n$. Eigenvalues of the matrix $\overline{A}_j$ are denoted by $\overline{\lambda}_{ji_j}$. We introduce the convention $\lambda_{ji_j}{}^{[1]}=\lambda_{ji_j}$ and $\lambda_{ji_j}{}^{[0]}=\overline{\lambda}_{ji_j}$.

According to Theorem 2.10 we have that for regular graphs relations $\overline{\lambda}_{j1}=m_j-1-\lambda_{j1}$ and $\overline{\lambda}_{ji_j}=-1-\lambda_{ji_j}$ $(i_j=2, \ldots, m_j)$ hold. The eigenvector u_{ji_j} is simultaneously the eigenvector for $\overline{\lambda}_{ji_j}$ in the matrix $\overline{A}_j$, what is proved in **[77]**.

Theorem 5.4. *If $G_1, \ldots, G_n$ are regular graphs, the spectrum of the graph $G=f(G_1, \ldots, G_n)$ is the set $\{\Lambda_{i_1,\ldots,i_n} \mid i_i=1, \ldots, m_j,\ j=1, \ldots, n\}$, where*

$$(5.5) \qquad \Lambda_{i_1,\ldots,i_n}=\sum_{\beta\in F}\lambda_{1i_1}{}^{[\beta_1]}\cdots\lambda_{ni_n}{}^{[\beta_n]}.$$

The eigenvector $u_{i_1,\ldots,i_n}=u_{1i_1}\otimes\cdots\otimes u_{ni_n}$ corresponds to the eigenvalue $\Lambda_{i_1,\ldots,i_n}$.

This theorem is proved in **[20]**.

5.3. Connectedness of NEPS and of Boolean functions

We shall investigate the connectedness of NEPS and of Boolean functions by using Theorems 4.17 and 4.18.

Consider the NEPS of graphs $G_1, \ldots, G_n$ each containing at least one edge. Indices $r_1, \ldots, r_n$ of graphs are then positive. By analysis of expression (5.4) we see that the index of NEPS is obtainable from (5.4) if we put $i_1=$ $=i_2=\cdots=i_n=1$, i. e., according to the accepted convention, $\lambda_{ji_j}=\lambda_{j1}=r_j$ $(j=1, \ldots, n)$. Thus, for index r of NEPS we have

$$(5.6) \qquad r=\Lambda_{1,\ldots,1}=\sum_{\beta\in B}r_1{}^{\beta_1}\cdots r_n{}^{\beta_n} \qquad (>0).$$

We shall consider only the NEPS with the basis B, for which there exists in B for every $j(j=1, \ldots, n)$ at least one n-tuple $(\beta_1, \ldots, \beta_n)$ whith $\beta_j=1$. We will denote this condition by (D).

If $G_1, \ldots, G_n$ are connected graphs, positive eigenvectors $x_1, \ldots, x_n$ belong to indices $r_1, \ldots, r_n$. It can be easily verified (see also **[20]**) that the eigenvector $x = x_1 \otimes \cdots \otimes x_n$ belongs to index r of NEPS. Thus, x is a positive vector too.

According to Theorem 4.18, we then have that the number of components of NEPS is equal to the multiplicity of index r. Hence, it is necessary to investigate whether or not $\Lambda_{i_1, \ldots, i_n}$ is equal to r for some n-tuple $(i_1, \ldots, i_n)$ different from $(1, \ldots, 1)$. Thus, it is necessary for this that at least for one j $(j = 1, \ldots, n)$ the relation $|\lambda_{ji_j}| = |\lambda_{j1}| = r_j$ holds. Since $G_1, \ldots, G_n$ are connected graphs, its indices are one-fold eigenvalues and the above equality can be satisfied only if $\lambda_{ji_j} = -r_j$. According to Theorem 4.3 we than have that G_j is a bipartite graph.

Hence, the possible unconnectedness of the NEPS of connected graphs, each containing at least one edge, appears as a consequence of the bipartity of these graphs. However, the bipartity does not always cause unconnectedness. The structure of the considered NEPS has a certain influence too.

By further analysis we see that the requested n-tuple of indices $i_1, \ldots, i_n$ must be such, that for every $i_j \neq 1$ the graph G_j is bipartite, i. e., that $\lambda_{ji_j} = -r_j$, and that every summand in (5.4) contains an even number of quantities λ_{ji_j} $(i_j \neq 1)$.

In order to formulate the theorem precising the conditions for connectedness (and later also for bipartity) of NEPS we introduce the following definition.

Definition 5.4. *A function in several variables is called even (odd) with respect to a given non-empty subset of variables if the function does not change its value (it changes only its sign) when variables from the considered subset change simultaneously their sign. The function is even (odd) if at least one non-empty subset of variables exists with respect to which the function is even (odd).*

According to the above facts we get the following theorem.

Theorem 5.5. *Let $G_1, \ldots, G_n$ be connected graphs each containing at least one edge. Suppose also that $G_{i_1}, \ldots, G_{i_s}$ $(\{i_1, \ldots, i_s\} \subset (\{1, \ldots, n\})$ are bipartite. NEPS with the basis B, satisfying condition (D), of graphs $G_1, \ldots, G_n$ is connected graph if and only if the function*

$$\sum_{\beta \in B} x_1^{\beta_1} \cdots x_n^{\beta_n} \tag{5.7}$$

is even w. r. t. none of non-empty subsets of the set $L = \{x_{i_1}, \ldots, x_{i_s}\}$. In the case of unconnectedness the number of components is equal to the multiplicity of the index of NEPS.

The idea of this theorem may be found in **[15]**. In **[14]** the following theorem and a sketch of its proof is given. On the basis of Theorem 5.5 we now give a complete proof.

Theorem 5.6. *Let $G_1, \ldots, G_n$ be connected graphs each containing at least two vertices. p-sum of these graphs is a connected graph if, and only if, one of the following conditions holds: 1° p is equal to n and at most one of the graphs is bipartite; 2° p is odd and less than n; 3° p is even and less than n, where at least one of the graphs $G_1, \ldots, G_n$ is not bipartite. If p is equal to n and exactly*

$l\,(l>1)$ of the graphs $G_1, \ldots, G_n$ are bipartite, the p-sum has 2^{l-1} components. If p is even and less than n and if all the graphs $G_1, \ldots, G_n$ are bipartite, p-sum has two components.

Proof. Consider the function (5.7). For the p-sum this is the elementary symmetric function of order p of variables $x_1, \ldots, x_n$, i. e.,

$$\sum_{\{\alpha_1, \ldots, \alpha_p\}} x_{\alpha_1} \cdots x_{\alpha_p}, \tag{5.8}$$

where the summation is made over all combinations $\{\alpha_1, \ldots, \alpha_p\}$ of the set $\{1, \ldots, n\}$. The number of elements of L (see Theorem 5.5) is equal to l.

If $p=n$, (5.8) reduces to $x_1 \cdots x_n$. If $l>1$ this function is even w. r. t. some non-empty subsets of L. The number of components is greater than the number of such subsets by 1. Thus, p-sum has $1+\binom{l}{2}+\binom{l}{4}+\cdots=2^{l-1}$ components.

Under condition 2° the evenness of (5.8) does not depend on l; the function is not even. In order to prove this fact, we assume, on the contrary, that there exists a non-empty subset S of L w. r. t. which (5.8) is even. If S exists, there exists also the term $x_{i_1} \cdots x_{i_p}$ from (5.8) containing an even number of variables from S. Since p is odd, this term contains at least one variable not belonging to S. Let, for example, $x_{i_1} \in S$ and $x_{i_p} \notin S$. Since $p<n$, there exists at least one variable x_j not contained in $x_{i_1} \cdots x_{i_p}$. W. r. t. S the terms $x_{i_1}, \ldots, x_{i_{p-1}} x_j$ and $x_j x_{i_2} \cdots x_{i_p}$ are not even, namely the first in case $x_j \in S$ and the latter in case $x_j \notin S$. In both cases we see that (5.8) cannot be an even function under condition 2°.

If condition 3° holds, we again assume that (5.8) is even w. r. t. to some subset S of L and consider the term $x_{i_1} \cdots x_{i_p}$ which contains $2k$ (k natural number) elements from S. If $2k<p$, we can prove in the same way as earlier that (5.8) contains a term not even w. r. t. S. If, however, $2k=p$, consider the term $x_j x_{i_2} \cdots x_{i_p}$, where x_j is not contained in $x_{i_1} \cdots x_{i_p}$. Since (5.8) is, according to the assumption, even, it must be $x_j \in S$. Hence, $x_1, \ldots, x_n \in S$ and this is in contradiction with the fact that at least one of graphs $G_1, \ldots, G_n$ is not bipartite (condition 3°).

Finally, if p is even and less than n and if $l=n$ the function (5.8) is even w. r. t. L. The multiplicity of the index of the p-sum is then equal to 2.

This completes the proof of the theorem.

This theorem represents an amalgamation and a generalization of some particular results, as was pointed out in **[14]**. We add that the connectedness of sum was investigated in **[76]**, **[1]**, **[97]**. Apart from the papers just adduced and the ones in **[14]** the paper **[73]** deals also with connectedness of several binary operations on graphs. In the quoted papers the connectedness was investigated directly, by proving the existence of a path between two arbitrary vertices of the considered graph.

We proceed to the investigation of the connectedness of the Boolean function. We shall first prove a lemma.

Lemma 5.1. *The arbitrary Boolean function $G=f(G_1, \ldots, G_n)$ of regular graphs $G_1, \ldots, G_n$ is a regular graph.*

Proof. According to (5.5) the index r of G is given by

$$r=\Lambda_{1,\ldots,1}=\sum_{\beta\in F}\lambda_{11}^{[\beta_1]}\cdots\lambda_{n1}^{[\beta_n]}=\sum_{\beta\in F}r_1^{[\beta_1]}\cdots r_n^{[\beta_n]}, \tag{5.9}$$

with convention $r_j^{[1]}=r_j$, $r_j^{[0]}=\overline{r_j}=m_j-1-r_j\ (j=1,\ldots,n)$. The eigenvector $v=v_1\otimes\ldots\otimes v_n$, where $v_1,\ldots,v_n$ are eigenvectors of indices $r_1,\ldots,r_n$ of graphs $G_1,\ldots,G_n$, corresponds to r. Since $G_1,\ldots,G_n$ are regular graphs, vectors $v_1,\ldots,v_n$ have all components equal to 1, and thus this property has also the vector v. According to Theorem 4.16 we get the statement of the lemma.

We consider primarily the case when for all $j=1,\ldots,n$ among n-tuples of F at least one n-tuple exists for which $\beta_j=1$ and at least another n-tuple for which $\beta_j=0$. Denote this condition by (E). Then, there exist in (5.5) at least one term containing λ_{jij} and at least another term containing $\overline{\lambda_{jij}}$.

Consider the function

$$\sum_{\beta\in F}x_1^{\beta_1}\cdots x_n^{\beta_n}. \tag{5.10}$$

In **[20]** the following theorem is proved:

Theorem 5.7. *Let $G_1,\ldots,G_n$ be regular connected graphs, each containing at least one edge. Suppose also that neither of $G_1,\ldots,G_n$ is complete. Apart from that, let graphs $G_{j_1},\ldots,G_{j_s}$ ($\{j_1,\ldots,j_s\}\subset\{1,\ldots,n\}$) be bicomplete. The Boolean function $G=f(G_1,\ldots,G_n)$, satisfying condition (E), is a connected graph if, and only if, the non-empty subset of variables $x_{j_1},\ldots,x_{j_s}$ with respect to which the function* (5.10) *is even, does not exist.*

The proof is based on an analysis of the expression (5.5), i. e., on determination of the multiplicity of the index (5.9).

Example 5.1. For the disjunction of two graphs we have $F=\{(1,0),\ (0,1),\ (1,1)\}$, and the function (5.10) has the form $x_1+x_2+x_1x_2$. The function satisfies condition (E) and is not even. Therefore, disjunction of regular connected graphs, which are incomplete and which contain each at least one edge, is a connected graph.

Example 5.2. Consider the negation of the exclusive disjunction. Here is $F=\{(1,1),\ (0,0)\}$ and (5.10) becomes x_1x_2+1. This function is even w. r. t. $\{x_1x_2\}$ and $f(G_1,G_2)$ is an unconnected graph if G_1 and G_2 are bicomplete. The following fact is of interest. The product of bicomplete (in general, bipartite) graphs is an unconnected graph (see, for example, **[35]**). $f(G_1,G_2)$ contains all the edges of the product $G_1\times G_2$ and some others, but nevertheless it remains an unconnected graph. Note that $G_1\times G_2$ is the corresponded NEPS to $f(G_1,G_2)$.

We see in general that the connectedness of the NEPS corresponded to a Boolean function depends on the evenness of the function (5.10) in the same way as the connectedness of the Boolean function. So we have the following theorem.

Theorem 5.8. *Under conditions of Theorem* 5.7 *the Boolean function and to it corresponded* NEPS *are either both connected or both unconnected graphs.*

We shall now analyse the connectedness of the Boolean function in a more general case.

Let the condition (E) still hold, but let some graphs $G_1,\ldots,G_n$ be unconnected. However, as it is known, at least one of G_j and $\overline{G_j}$ is a connected graph. Considerations similar to the preceding ones lead to the conclusion that G_j can affect the connectedness of the Boolean function only if one of the graphs G_j and $\overline{G}_j$ is bicomplete. If G_j is unconnected it affects the connectedness only if $\overline{G}_j$ is bicomplete, i. e., if G_j has two components with the same number of vertices, each being a complete graph. It can easily be seen that in the latter case, due to condition (E), the possible bipartity of G_j has no influence. (The sole case of this kind is when G_j has, as components, two complete graphs, each containing two vertices.)

Thus, if condition (E) holds, if each of regular graphs $G_1, \ldots, G_n$ contains at least one edge and if none of them is a complete graph, then only bicomplete graphs and graphs having two components with the same number of vertices each of which is a complete graph, have an influence on the connectedness of the Boolean function.

Now let the condition (E) not hold.

If for any $j\,(j=1, \ldots, n)$ (5.5) contains in all summands the quantity λ_{ji_j}, then unconnectedness of the graph G_j directly causes the unconnectedness of the Boolean function. G_j affects the connectedness also then, when it is bipartite (connected), but, naturally, in this case an additional condition exists. Thus, the absence of the quantity $\overline{\lambda}_{ji_j}$ in (5.5) causes that, for unconnectedness, G_j must not be bicomplete but only a connected bipartite graph.

If all summands of the function (5.5) contain $\overline{\lambda}_{ji_j}$, all foregoing remarks are related to the complement $\overline{G}_j$ of G_j. Hence, if in this case G_j is not ∇ - prime, then $\overline{G}_j$ is unconnected and the Boolean function is an unconnected graph too. The supplementary consideration involves the case when G_j (which is regular of degree r_j and has m_j vertices) is ∇ - prime and has the following structure. The graph has two groups of vertices with the same number of vertices. Every vertex is adjacent to each vertex of its group and to exactly $r_j - \frac{m_j}{2} + 1$ vertices of the other group. Namely, in this and only in this case the component $\overline{G}_j$ is a connected bipartite graph.

We see that the connectedness of the Boolean function depends also on the evenness of a function, which we are going to define. This function is of the form (5.10). Instead of over the set of n-tuples F, the summation is made over the set H which is formed in the following way. In order to form H we depart from F and transform its n-tuples by use of the following operations:

1° If for any $j\,(j=1, \ldots, n)$ all n-tuples from F have on the j-th places 1's and the graph G_j is connected and bipartite, these 1's remain also in the n-tuples of H. If the graph has not the mentioned property, these 1's pass into the 0's.

2° If for any $j\,(j=1, \ldots, n)$ all n-tuples from F have on the j-th places 0's and the complement $\overline{G}_j$ of the graph G_j is connected and bipartite, then all mentioned 0's pass into 1's. If G_j does not have the above property, zeros remain on the j-th places in all n-tuples of H.

3° If for any $j\,(j=1, \ldots, n)$ at least one n-tuple from F has 1 on the j-th place and if at least one n-tuple has 0 on the same place, then: a) in the corresponding n-tuples of H nothing is to be changed on the j-th places if G_j is a bicomplete graph; b) 1's pass into 0's, and vice-versa, if G_j has two components with the same number of vertices, each representing a complete graph; c) in all n-tuples in the set H we put zeros if none of the preceding two cases occurred.

According to the fereging we have the following theorem.

Theorem 5.9. *Let $G_1, \ldots, G_n$ be regular incomplete graphs each containing at least an edge. The Bollean function $f(G_1, \ldots, G_n)$ is a connected graph if and only if the following conditions hold: 1° If for any $j\,(j=1, \ldots, n)$ all n-tuples from F contain* 1's *on the j-th place, the graph G_j is connected; 2° If for any $j\,(j=1, \ldots, n)$ all n-tuples from F contain* 0's *on the j-th place, the graph G_j is ∇-prime; 3° The function $\sum_{\beta\in H} x_1^{\beta_1} \cdots x_n^{\beta_n}$ is not even.*

Note that all properties of $G_1, \ldots, G_n$, which affect the connectedness of the Boolean function, can be determined by means of their spectra (see **3.** and **4.**).

5.4. Minimal functions

This Section is contained in paper **[15]**. We shall only give here some remarks.

REMARK 1. The problem of the minimal function, posed in [15] in relation to incomplete p-sum of graphs, can also be analogously posed for incomplete extended p-sum of graph.

REMARK 2. In a number of cases it was proved in [15], that no summand from the function (7) of [15] can be omitted without (7) becoming even. We note here that this is true also for $n=p-1$.

REMARK 3. Theorem of [15] and the idea for its proving is due to R. P. LUČIĆ.

5.5. Bipartity

In this Section we shall deduce the conditions under which the NEPS of connected graphs is a bipartite graph.

All components of a NEPS of connected graphs have the same index r. Thus, the number of components of such a NEPS is equal to the multiplicity of its index. NEPS is bipartite if, naturally, all its components are bipartite. According to Theorem 4.3. each component must then contain the number $-r$ in the spectrum. Since no component contains in the spectrum the number $-r$ with multiplicity greater than 1, it follows that a necessary and sufficient condition for bipartity of NEPS is that the numbers r and $-r$ have the same multiplicity in the spectrum of NEPS.

We see from (5.4) that the number $-r$ can exist in the spectrum of NEPS only if some of graphs are bipartite and if there exist subsets of variables $x_1, \ldots, x_n$ w. r. t. which the function (5.7) is odd. According to the foregoing facts we get the following theorem:

Theorem 5.10. *Let $G_1, \ldots, G_n$ be connected graphs, each containing at least one edge. Suppose also that $G_{i_1}, \ldots, G_{i_s}$ $(\{i_1, \ldots, i_s\}\subset\{1, \ldots, n\})$ are bipartite.* NEPS *with the basis B, satisfying condition (D), of graphs $G_1, \ldots, G_n$ is bipartite if and only if the number of non-empty subsets of the set $L=\{x_{i_1}, \ldots, x_{i_s}\}$, w. r. t. which the function* (5.7) *is even, is smaller by 1 than the number of such subsets w. r. t. which it is odd.*

This theorem represents the basis for proving the following theorem, mentioned in **[14]**, precising the conditions under which the p-sum is a bipartite graph.

Theorem 5.11. *Let $G_1, \dots, G_n$ be connected graphs each containing at least two vertices. p-sum of these graphs is a bipartite graph if, and only if, one of the following condition holds:* 1° *p is equal to n and at least one of the graphs $G_1, \dots, G_n$ is bipartite;* 2° *p is odd and less than n and all the graphs $G_1, \dots, G_n$ are bipartite.*

Proof. Let $p=n$. The function (5.7) i. e. (5.8) is then of the form $x_1 \cdots x_n$. If $L=\varnothing$ (see Theorem 5.10), p-sum is not bipartite. Let L contain $l\,(l \geq 1)$ elements. Then the function $x_1 \cdots x_n$ is even w. r. t. exactly $\binom{l}{2}+\binom{l}{4}+\cdots=2^{l-1}-1$ non-empty subsets of L and is odd w. r. t. exactly $\binom{l}{1}+\binom{l}{3}+\cdots=2^{l-1}$ such subsets. According to Theorem 5.10, p-sum is then bipartite.

Let p be odd and less than n. The function (5.8) is then not even (Theorem 5.6.) It is odd only w. r. t. all variables because if it were odd w. r. t. a proper subset of variables, then one summand among the summands of (5.8) would exist, containing an even number of variables from the same subset, which is in contradiction with the assumption of oddity of (5.8). Thus, for the bipartity of p-sum in this case it is necessary (and sufficient) that all the graphs $G_1, \dots, G_n$ are bipartite.

Finally, if p is even and less than n, using similar reasoning, we see that (5.8) cannot be odd.

This completes the proof of the Theorem.

5.6. The number of walks

As it was already said in **[16]**, the results from **[16]** can be extended to some operations on graphs for which adjacency matrices are expressed in terms of normal summands. Considerations similar to that of **[16]** give the following theorem:

Theorem 5.12. *Let $N_k^j=\sum_{i_j} C_{ji_j}\lambda_{ji_j}^k \; (j=1, \dots, n)$ denote the number of length k for G_j.* NEPS *with the basis B of graphs $G_1, \dots, G_n$ contains*

$$N_k=\sum_{i_1, \dots i_n} C_{1i_1}\cdots C_{ni_n}\Big(\sum_{\beta\in B}\lambda_{1i_1}^{\beta_1}\cdots\lambda_{ni_n}^{\beta_n}\Big)^k$$

walks of length k.

EXAMPLE 5.3. For some chessfigures, the graphs of their moving on the two or many--dimensional chessboards can be expressed in form of NEPS of graphs of its moving on one--dimensional boards. So it is mentioned in **[1]**, that the graph of rook's move on a two-dimensional square board is equal to sum of graphs of rook's move on a one-dimensional board with itself. (This is mentioned in **[1]** for the case when the rook moves only for a field in a move, but it is obviously that this holds in general case).

In **[18]** the king's moving on a two-dimensional square board was considered. From this consideration it follows that everything that has been said for the rook is valid also for the king, only the sum of graphs is to be interchanged with the strong product (NEPS with the basis containing all possible n-tuples) of graphs.

If the rook moves only for a field, the corresponding graph of its moving on one dimensional board is not different from the graph of king's move on the same board. The number of walks of length k was determined in [**18**]. Then according to Theorem 5.12 we have for the number of ways that a rook (if it moves along for a field) i. e. a king makes a series of k moves on s-dimensional chessboard of dimensions $n_1 \times \cdots \times n_s$:

$$N_k = \sum_{i_1, \ldots, i_s} C_{1i_1} \cdots C_{si_s} (\lambda_{1i_1} + \cdots + \lambda_{si_s})^k, \tag{5.11}$$

i. e.

$$N_k = \sum_{i_1, \ldots, i_s} \left(\prod_{j=1}^{s} C_{ji_j}\right) \left(\prod_{j=1}^{s} (\lambda_{ji_j} + 1) - 1\right)^k, \tag{5.12}$$

$$i_j = 1, \ldots, n_j \quad (j = 1, \ldots, s),$$

where

$$C_{ji_j} = \frac{2}{n_j + 1} \operatorname{cotg}^2 \frac{2 i_j - 1}{n_j + 1} \frac{\pi}{2}, \qquad \lambda_{ji_j} = 2 \cos \frac{2 i_j - 1}{n_j + 1} \pi.$$

In [**18**] a special case of the result (5.12) was obtained. This is the case $s = 2$, $n_1 = n_2 = n$, which corresponds to king's moving on the $n \times n$ board.

REFERENCES[1]

1. O. ABERTH: *On the sum of graphs*, Rev. Franc. Rech. Operat. **8** (1964), 353—358.
2.* G. A. BAKER: *Drum shapes and isospectral graphs.* J. Math. Phys. **7** (1966), 2238—2242.
3. C. BERGE: *Théorie des graphes et ses applications*, deuxième édition. Paris 1963.
4. R. C. BOSE: *Strongly regular graphs, partial geometries and partially balanced designs.* Pacific J. Math. **13** (1963), 389—419.
5.* R. C. BOSE, R. LASKAR: *Eigenvalues of the adjacency matrix of tetrahedral graphs.* Aequationes Math. **4** (1970), 37—43.
6.* W. BROWN: *On the non-existence of a type of regular graphs of girth* 5. Canad. J. Math. **19** (1967), 644—648.
7.* R. H. BRUCK: *Finite nets II. Uniqueness and embedding.* Pacific J. Math. **13** (1963), 421—457.
8. L. C. CHANG: *The uniqueness and non-uniqueness of the triangular association scheme.* Sci. Record **3** (1959), 604—613.
9. L. C. CHANG: *Association schemes of partially balanced block designs with parameters* $v = 28$, $n_1 = 12$, $n_2 = 15$ *and* $p_{11}^2 = 4$. Sci. Record **4** (1960) ,12—18.
10.* L. COLLATZ, U. SINOGOWITZ: *Spektren endlicher Grafen.* Abh. Math. Sem. Univ. Hamburg **21** (1957), 63—77.
11. W. S. CONNOR: *The uniqueness of the triangular association scheme.* Ann. Math. Statist. **29** (1958), 262—266.
12.* D. CVETKOVIĆ: *Bihromatičnost i spektar grafa.* Matematička biblioteka № 41, Beograd 1969, 193—194.
13.* D. CVETKOVIĆ: *Spectrum of the graph of n-tuples.* Univ. These Publications № **274**—№ **301** (1969), 91—95.

[1] Papers, wherein the spectra or characteristic polynomials of graphs appear explicitely, are denoted by an asterisk. I got familiar with the papers [**25**], [**26**], [**49**], [**50**], [**51**], [**57**], [**62**] only upon the completion of the Serbo-Croatian version of my Dissertation. Papers [**21**], [**22**], [**54**] have also been written in that period.

14.* D. Cvetković: *Connectedness of the p-sum of graphs.* Ibid. № **274**—№ **301** (1969), 96—99.

15.* D. M. Cvetković, R. P. Lučić: *A new generalization of the concept of the p-sum of graphs.* Ibid. № **302**—№ **319** (1970), 67—71.

16.* D. M. Cvetković: *A note on paths in the p-sum of graphs.* Ibid. № **302**—№ **319** (1970), 49—51.

17.* D. M. Cvetković: *The generating function for variations with restrictions and paths of the graph and self-complementary graphs.* Ibid. № **320**—№ **328** (1970), 27—34.

18.* D. Cvetković: *Die Zahl der Wege eines Grafen.* Glasnik Mat. Ser. III, **5** (**25**) (1970), 205—210.

19.* D. M. Cvetković: *New characterizations of the cubic lattice graph.* Publ. Inst. Math. (Beograd) **10** (**24**) (1970), 195—198.

20.* D. Cvetković: *Booleove operacije nad grafovima — spektar i povezanost.* (summary), V kongres mat. fiz. i astr. Jugoslavije, Ohrid 1970, 44, *The Boolean operations on graphs.* To appear.

21.* D. M. Cvetković: *The spectral method for determining the number of trees.* Publ. Inst. Mat. (Beograd) **11** (**25**) (1971), 135-141.

22.* D. M. Cvetković: *Inequalities obtained on the basis of the spectrum of the graph.* To appear.

23. K. Čulik: *Zur Theorie der Graphen.* Časopis Pěst. Mat. **83** (1958), 133—155.

24.* D. Ž. Djoković: *Isomorphism problem for a special class of graphs.* Acta Math. Acad. Sci. Hungar. **21** (1970), 267—270.

25.* M. Doob: *Graphs with a small number of distinct eigenvalues.* Ann. New York Acad. Sci. **175** (1970), № 1, 104—110.

26.* M. Doob: *On characterizing certain graphs with four eigenvalues by their spectra.* Linear Algebra and Appl. **3** (1970), 461—482.

27.* A. L. Dulmage, N. S. Mendelsonh: *Graphs and matrices, (Graph theory and theoretical physics).* London — New York 1967, p. p. 167—227.

28.* B. Elspas, J. Turner: *Graphs with circulant adjacency matrices.* J. Combinatorial Theory **9** (1970), 297—307.

29.* H. -J. Finck, G. Grohmann: *Vollständiges Produkt, chromatische Zahl und characteristisches Polynom regularer Graphen I.* Wiss. Z. Techn. Hochsch. Ilmenau **11** (1965), 1—3.

30.* H. -J. Finck: *Vollständiges Produkt, chromatische Zahl und characteristisches Polynom regulärer Graphen II.* Ibid. **11** (1965), 81—87.

31.* M. Fisher: *On hearing the shape of a drum.* J. Combinatorial Theory **1** (1966), 105—125.

32. F. R. Gantmacher: *The theory of matrices I, II.* New York 1960.

33.* J. M. Goethals, J. J. Seidel: *Strongly regular graphs derived from combinatorial designs.* Canad. J. Math. **22** (1970), 597—614.

34.* F. Harary: *The determinant of the adjacency matrix of a graph.* SIAM Rev. **4** (1962), 202—210.

35. F. Harary, G. W. Wilcox: *Boolean operations on graphs.* Math. Scand. **20** (1967), 41—51.

36.* A. J. Hoffman: *On the uniqueness of the triangular association scheme.* Ann. Math. Statist. **31** (1960), 492—497.

37.* A. J. Hoffman: *On the exceptional case in a characterization of the arcs of a complete graph.* IBM J. Res. Develop. **4** (1960), 487—496.

38.* A. J. Hoffman, R. R. Singleton: *On Moore graphs with diameters 2 and 3.* Ibid. **4** (1960), 497—504.

39.* A. J. Hoffman: *On the polynomial of a graph.* Amer. Math. Monthly **70** (1963), 30—36.

40.* A. J. Hoffman, M. H. McAndrew: *The polynomial of a directed graph.* Proc. Amer. Math. Soc. **16** (1965), 303—309.

41.* A. J. HOFFMAN, D. K. RAY-CHAUDHURI: *On the line graph of a finite plane.* Canad. J. Math. **17** (1965), 687—694.

42.* A. J. HOFFMAN: *On the line graph of a projective plane.* Proc. Amer. Math. Soc. **16** (1965), 297—302.

43.* A. J. HOFFMAN, D. K. RAY-CHAUDHURI: *On the line graph of a symmetric balanced incomplete block designs.* Trans. Amer. Math. Soc. **116** (1965), № 4, 238—252.

44.* A. J. HOFFMAN: *Some recent results on spectral properties of graphs, (Beiträge zur Graphentheorie).* Leipzig 1968, p. p. 75—80.

45.* A. J. HOFFMAN: *The change in the least eigenvalue of the adjacency matrix of a graph under imbeding.* SIAM J. Appl. Math. **17** (1969), 664—671.

46.* A. J. HOFFMAN: *The eigenvalues of the adjacency matrix of a graph. (Combinatorial mathematics and its applications),* Chapel Hill 1969, p. p. 578—584.

47.* A. J. HOFFMAN, D. K. RAY-CHAUDHURI: *On a spectral characterization of regular line graphs.* To appear.

48.* A. J. HOFFMAN, A. M. OSTROWSKI: *On the least eigenvalue of a graph of large minimum valence containing a given graph.* Linear Algebra and Appl. To appear.

49.* A. J. HOFFMAN: *On eigenvalues and colouring of graphs I.* Proc. Advanc. Seminar on Graph Theory, Army Mathematics Center, University of Wisconsin.

50.* A. J. HOFFMAN, L. HOWES: *On eigenvalues and colourings of graphs II.* Ann. New York, Acad. Sci. **175** (1970), № 1, 238—242.

51.* L. HOWES: *On subdominantly bounded graphs — summary of results. (Recent Trends in Graph Theory),* Berlin — Heidelberg — New York 1971, p. p. 181—183.

52.* H. HUTSCHENREUTER: *Einfacher Beweis des Matrix-Gerüst-Satzes der Netzwerktheorie.* Wiss. Z. Techn. Hochsch. Ilmenau **13** (1967), 403—404.

53.* P. W. KASTELEYN: *Graph theory and crystal physics, (Graph theory and theoretical physics).* London — New York 1967, p. p. 43—110.

54.* L. L. KRAUS, D. M. CVETKOVIĆ: *Evaluation of a lower bound for the chromatic number of the complete product of graphs.* These Publications, in print.

55.* W. KUICH, N. SAUER: *On the existence of certain minimal regular n-systems with given girth, (Proof Techniques in Graph Theory).* New York — London 1969, p. p. 93—101.

56.* R. LASKAR: *Eigenvalues of the adjacency matrix of cubic lattice graphs.* Pacific J. Math. **29** (1969), 623—629.

57.* D. R. LICK: *A class of partition numbers. (Recent Trends in Graph Theory).* Berlin — Heidelberg — New York 1971, p. p. 184—190.

58.* J. H. VAN LINT, J. J. SEIDEL: *Equilateral point sets in elliptic geometry.* Nederl. Akad. Wetensch. Proc. Ser. A. **69** (1966), 335—348.

59. M. MARCUS, H. MINC: *A survey of matrix theory and matrix inequalities* Boston 1964.

60.* R. B. MARIMONT: *System connectivity and matrix properties.* Bull. Math. Biophys **31** (1969), 255—274.

61. D. J. MILLER: *The categorical product of graphs.* Canad. J. Math. **20** (1968), 1511—1521.

62.* J. MITCHEM: *On extremal partitions of graphs.* Thesis, Michigan 1970.

63. W. O. J. MOSER, M. ABRAMSON: *Enumeration of combinations with restricted differences and cospan.* J. Combinatorial Theory **7** (1969), 162—170.

64.* A. MOWSHOWITZ: *Entropy and complexity of graphs III, Graphs with prescribed information content.* Bull. Math. Biophys. **30** (1968), 387—414.

65.* A. MOWSHOWITZ: *The group of a graph whose adjacency matrix has all distinct eigenvalues, (Proof Techniques in Graph Theory).* New York — London p. p. 109—110.

66.* E. A. NORDHAUS: *A class of strongly regular graphs, (Proof Technique in Graph Theory).* New York — London 1969, p. p. 119—123.

67. O. ORE: *Theory of graphs.* Providence 1962.

68.* M. PETERSDORF: *Über Zusammenhänge zwischen Automorphismengruppe, Eigenwerten und vorderen Teilern eines Graphen und deren Verwendung zur Interpretation und Lösung gewisser Probleme der Analysis und Algebra.* Thesis, Ilmenau 1969.

69.* M. PETERSDORF, H. SACHS: *Über Spektrum, Automorphismengruppe und Teiler eines Graphen.* Wiss. Z. Techn. Hochsch. Ilmenau **15** (1969), 123—128.

70.* M. PETERSDORF, H. SACHS: *Spektrum und Automorphismengruppe eines Graphen,* (*Berichte der Tagung über kombinatorische Strukturen.*) Balatonführed, Ungarn, August 1969.

71.* J. PONSTEIN: *Self-avoiding paths and the adjacency matrix of a graph.* SIAM J. Appl. Math. **14** (1966), 600—609.

72.* D. K. RAY-CHAUDHURI: *Characterization of line graphs.* J. Combinatorial Theory **3** (1967), 201—214.

73. K. B. REID: *Connectivity in products of graphs.* SIAM Rev. **18** (1970), 645—651.

74. G. RINGEL: *Selbstkomplementäre Graphen.* Arch. Math. (Basel). **14** (1963), 354—358.

75. D. E. RUTHERFORD: *Some continuant determinants arising in physics and chemistry.* Proc. Roy. Soc. Edinburg Sect. A. **62** (1947), 229.

76. G. SABIDUSSI: *Graph multiplication.* Math. Z. **72** (1960), 446—457.

77.* H. SACHS: *Über selbstkomplementäre Graphen.* Publ. Math. (Debrecen) **9** (1962), 270—288.

78.* H. SACHS: *Beziehungen zwischen den in einem Graphen enthaltenen Kreisen und seinem charakteristischen Polynom.* Ibid. **11** (1963), 119—134.

79.* H. SACHS: *Über Teiler, Faktoren und characteristische Polynome von Graphen,* Teil I. Wiss. Z. Techn. Hochsch. Ilmenau **12** (1966), 7—12.

80.* H. SACHS: *Über Teiler, Faktoren und characteristische Polynome von Graphen,* Teil II. Ibid. **13** (1967), 405—412.

81. F. SCHEID: *Some packing problem.* Amer. Math. Monthly **67** (1960), 231—235.

82.* V. SEDLAČEK: *O incidenčnich orientovaných grafů.* Časopis Pěst. Mat. **84** (1959), 303—316.

83.* J. J. SEIDEL: *Strongly regular graphs of L_2 - type and of triangular type.* Indagationes Math. **29** (1967), 188—196.

84.* J. J. SEIDEL: *Strongly regular graphs with* $(-1, 1, 0)$*—adjacency matrix having eigenvalue* 3. Linear Algebra and Appl. **1** (1968), 281—298.

85.* J. J. SEIDEL: *Strongly regular graphs,* (*Recent Progress in Combinatorics.*) New York — London 1969 pp. 185—198.

86. S. S. SHRIKHANDE: *On a characterization of the triangular association scheme.* Ann. Math. Statist **30** (1959), 39—47.

87. S. S. SHRIKHANDE: *The uniqueness of the L_2 association scheme.* Ibid. **30** (1959), 781—798.

88.* R. SINGLETON: *On minimal graphs of maximum even girth.* J. Combinatorial Theory **1** (1966), 306—332.

89.* M. TUERO: *A contribution to the theory of cyclic graphs.* Matrix Tensor Quart. **11** (1961), 74—80.

90.* J. TURNER: *Point-symmetric graphs with a prime number of points.* J. Combinatorial Theory **3** (1967), 136—145.

91.* J. TURNER: *Generalized matrix functions and the graph isomorphisme problem.* SIAM J. Appl. Math. **16** (1968), 520—526.

92.* T. H. WEI: *The algebraic foundations of ranking theory.* Thesis, Cambridge 1952.

93. P. M. WEICHSEL: *The Kronecker product of graphs.* Proc. Amer. Math. Soc. **13** (1962), 47—52.

94.* H. S. WILF: *The eigenvalues of a graph and its chromatic number.* J. London Math. Soc. **42** (1967), 330—332-

95.* А. БЕЦЕ: *О коэффициентах характеристического полинома графа.* Латв. матем. ежегодник 3 (1968), 75—80.

96.* Е. В. ВАХОВСКИ: *О характеристических числах матрицах соседства для неособенних графов.* Сибир. матем. журн. **6** (1965), № 1, 44—49.

97. В. Г. Визинг: *Декартово произведение графов.* Вычисл. системы **9** (1963), 30—43.

98.* А. К. Кельманс: *О числе деревьев графа I.* Автоматика и телемеханика **26** (1965), 2194—2204.

99.* А. К. Кельманс: *О числе деревьев графа II.* Автоматика и телемеханика 1966, № 2, 56—65.

100.* А. К. Кельманс: *О свойствах характеристического многочлена графа,* (*Кибернетику на службу комунизму*). Москва-Ленинград 1967, *p.p.* 27—41.

101.* А. К. Кельманс: *О „разрешающей способности" характеристичвского многочлена матрицы проводимости графа.* Unpublished.

102.* Л. М. Лихтенбаум: *Характеристические числа неособенного графа.* Труды 3-го Всес. матем. съезда, том **1**, 1956, 135—136.

103.* Л. М. Лихтенбаум: *Теорема двойственности для неособенных графов.* Усп. мат. наук **13** (1958), № 5, 185—190.

104.* Л. М. Лихтенбаум: *Следи степеней матриц соседства вершин и ребер неособенного графа.* Изв. высш. учебн. завед. математика **5** (1959), 154—163.

105.* Л. М. Лихтенбаум: *Новые теоремы о графах.* Сибир. матем. журн. **3** (1962), 561—568.

CONTENTS