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347. SOME CONTRIBUTIONS TO THE MEAN VALUE THEOREM*

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0. In his paper **[1]** J. KARAMATA has generalized the theorem on mean values, replacing the existance of the first derivative by the existance of the left and right derivatives in the interval in question. Using these results, V. VUČKOVIĆ **[2]** generalized, in the same way all the important consequences of LAGRANGE's theorem. All these theorems from **[1]** and **[2]** reduce to the usual theorems of that type (i.e. the theorems exposed in **[3]**) if one supposes that the derivative exists. The mentioned theorems refer to real functions.

In paper [4] S. REICH using DINI's derivatives, proved the following theorem:

Let f be a function with the following properties: 1) f is continuous in $[a, b]$, 2) the four DINI derivatives of f are finite in (a, b), 3) $f(a)=f(b)$. Then either (a) there exists c in (a, b) such that $f^+(c)\leqq 0$ and $f_-(c)\geqq 0$, or (b) there exists d in (a, b) such that $f_+(d)\geqq 0$ and $f^-(d)\leqq 0$.

The proof of this theorem is left out in paper [4], since it can be found in **[5]**.

In this paper we shall generalize FERMAT's, ROLLE's, LAGRANGE's and CAUCHY's theorems analogously to **[1]** and **[2]**. These generalizations include the mentioned theorem from **[4]**, but in a new form.

1. In this paper we shall use the following definitions and results:

1.1. Let $x_k\,(k=1, \ldots, n)$ be real numbers with $a=\min(x_1, \ldots, x_n)$ and $b=\max(x_1, \ldots, x_n)$. Then any number $t\in[a, b]$ can be represented in the form

$$t=\sum_{k=1}^{n} p_k x_k$$

where $p_k\geqq 0\,(k=1, \ldots, n)$ and where $\sum\limits_{k=1}^{n} p_k=1$.

1.2. In **[6]** one can find the following definition: If for a function f for some $x\in R$ there exist

$$\overline{\lim_{h\to 0+}}\, D(x, h),\ \underline{\lim_{h\to 0+}}\, D(x, h),\ \overline{\lim_{h\to 0-}}\, D(x, h),\ \underline{\lim_{h\to 0-}}\, D(x, h),$$

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then we say that f has the upper right, lower right, upper left, lower left derivative respectively in x, which we denote $D^+f(x)$, $D_+f(x)$, $D^-f(x)$, $D_-f(x)$, i.e. by D^+, D_+, D^-, D_-, if we know which point is considered. Besides, we have

$$D(x, h) = \frac{f(x+h)-f(x)}{h}.$$

1.3. If f and g are defined on $I \subset R$, then if the limit values

$$\overline{\lim_{x\to x_0}}\, f(x), \quad \underline{\lim_{x\to x_0}}\, f(x), \quad \lim_{x\to x_0} g(x)$$

exist when $x \to x_0$ through I, then

$$\overline{\lim_{x\to x_0}}\,(f(x)+g(x)) = \overline{\lim_{x\to x_0}}\, f(x) + \lim_{x\to x_0} g(x),$$

$$\underline{\lim_{x\to x_0}}\,(f(x)+g(x)) = \underline{\lim_{x\to x_0}}\, f(x) + \lim_{x\to x_0} g(x).$$

2. The following theorem is a generalization of FERMAT's theorem (see, for example [3]):

Theorem 1. *Let f be defined in $[a, b]$ and let f have the derivatives D^+, D_+, D^-, D_- in (a, b). If f reaches its highest (lowest) value for $r \in (a, b)$ then there are real numbers $p_k \geqq 0$ $(k=1, 2, 3, 4)$ where $p_1+p_2+p_3+p_4=1$, such that*

$$p_1 D^+f(r) + p_2 D_+f(r) + p_3 D^-f(r) + p_4 D_-f(r) = 0.$$

Proof. Let f reach its minimal value in $r \in (a, b)$. Then

$$f(r+h) - f(r) \geqq 0 \tag{1}$$

for all $h \neq 0$ and all $r+h \in (a, b)$. Let $h > 0$. Then

$$\frac{f(r+h)-f(r)}{h} \geqq 0$$

and, since D^+, D_+, D^-, D_- exist, by 1.2. we see that

$$D^+f(r) \geqq 0, \quad D_+f(r) \geqq 0.$$

For $h \leqq 0$, in a similar way we conclude

$$D^-f(r) \leqq 0, \quad D_-f(r) \leqq 0.$$

Since $D_- \leqq D^-$ and $D_+ \leqq D^+$, we have

$$D_-f(r) \leqq D^-f(r) \leqq 0 \leqq D_+f(r) \leqq D^+f(r)$$

and therefore the interval $[D_-f(r), D^+f(r)]$ contains the numbers 0, $D^-f(r)$, $D_+f(r)$. According to 1.1. we can write

$$0 = p_1 D^+f(r) + p_2 D_+f(r) + p_3 D^-f(r) + p_4 D_-f(r)$$

where $p_k \geqq 0$ $(k=1, 2, 3, 4)$ and $p_1+p_2+p_3+p_4=1$. This completes the proof.
The proof is analogous when f reaches its maximum in $r \in (a, b)$.

The mentioned theorem from [4] is a consequence of Theorem 1, and can be formulated in this way:

Theorem 2. *Let*

1) $f(x) \in C_{[a, b]}$; $f(a) = f(b)$,

2) *derivatives* D^+, D_+, D^-, D_- *exist for* f *in* (a, b).

Then there exist real numbers $p_k \geqq 0$ $(k = 1, 2, 3, 4)$ *with* $p_1 + p_2 + p_3 + p_4 = 1$, *so that*

$$p_1 D^+ f(r) + p_2 D_+ f(r) + p_3 D^- f(r) + p_4 D_- f(r) = 0$$

for some number $r \in (a, b)$.

This form of ROLLE's theorem is a consequence of Theorem 1 and of Theorem from [4] and is a generalization of the KARAMATA—VUČKOVIĆ type.

LAGRANGE's theorem is a direct corollary of Theorem 2.

Theorem 3. *Let*

1) $f(x) \in C_{[a, b]}$,

2) *the derivatives* D^+, D_+, D^-, D_- *exist for* f *in* (a, b).

Then there exist $r \in (a, b)$ *and real numbers* $p_k \geqq 0$ $(k = 1, 2, 3, 4)$ *with* $p_1 + p_2 + p_3 + p_4 = 1$, *so that*

$$\frac{f(b) - f(a)}{b - a} = p_1 D^+ f(r) + p_2 D_+ f(r) + p_3 D^- f(r) + p_4 D_- f(r).$$

Proof. Consider, as usual, the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \tag{2}$$

From (2) it follows

$$\frac{F(x+h) - F(x)}{h} = \frac{f(x+h) - f(x)}{h} - \frac{f(b) - f(a)}{b - a}$$

for $h \neq 0$. Using 1.3. we directly have

$$\begin{aligned} D^+ F(x) &= D^+ f(x) - \frac{f(b) - f(a)}{b - a}, \\ D_+ F(x) &= D_+ f(x) - \frac{f(b) - f(a)}{b - a}, \\ D^- F(x) &= D^- f(x) - \frac{f(b) - f(a)}{b - a}, \\ D_- F(x) &= D_- f(x) - \frac{f(b) - f(a)}{b - a} \end{aligned} \tag{3}$$

and, therefore, D^+, D_+, D^-, D_- exist for $F(x)$ in (a, b). Since $F(x)$ is continuous by the hypothesis in $[a, b]$ and since from (2) we see that $F(a) = F(b) = 0$ according to Theorem 2 there exists $r \in (a, b)$ and real numbers $p_k \geqq 0$ $(k = 1, 2, 3, 4)$ where $p_1 + p_2 + p_3 + p_4 = 1$, so that

$$p_1 D^+ F(r) + p_2 D_+ F(r) + p_3 D^- F(r) + p_4 D_- F(r) = 0. \tag{4}$$

Multiplying (3) by p_k, adding and using (4), the Theorem follows.

CAUCHY's theorem can be proved in following form:

Theorem 4. *Let*

1) $f(x),\ g(x) \in C_{[a,\, b]},\ g(a) \neq g(b)$,
2) *the derivatives* D^+, D_+, D^-, D_- *exist for* $f(x)$ *in* (a, b),
3) $g'_+(x)$ *and* $g'_-(x)$ *exist in* (a, b),
4) $0 \notin [g'_-, g'_+]$ *or* $0 \notin [g'_+, g'_-]$,

then there exists $r \in (a, b)$ *and real numbers* $p_k \geqq 0$ $(k = 1, 2, 3, 4)$, *with*

$$p_1 + p_2 + p_3 + p_4 = 1,$$

so that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{p_1 D^+ f(r) + p_2 D_+ f(r) + p_3 D^- f(r) + p_4 D_- f(r)}{(p_1+p_3)\, g'_-(r) + (p_2+p_4)\, g'_+(r)}.$$

Proof. Introduce the function

$$F(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} (g(x) - g(a)). \tag{5}$$

$F(x)$ is continuous in $[a, b]$, $F(a) = F(b) = 0$. From (5) it also follows

$$\frac{F(x+h)-F(x)}{h} = \frac{f(x+h)-f(x)}{h} - \frac{f(b)-f(a)}{g(b)-g(a)} \frac{g(x+h)-g(x)}{h}. \tag{6}$$

From (6) it directly follows

$$\begin{aligned} D^- F(x) &= D^- f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'_-(x), \\ D_- F(x) &= D_- f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'_-(x), \end{aligned} \tag{7}$$

by 1.3. In the same way we also get

$$\begin{aligned} D^+ F(x) &= D^+ f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'_+(x) \\ D_+ F(x) &= D_+ f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'_+(x) \end{aligned} \tag{8}$$

by 1.3. On the other hand, according to Theorem 2 there exists $r \in (a, b)$ and real numbers $p_k \geqq 0$ $(k = 1, 2, 3, 4)$ with $p_1 + p_2 + p_3 + p_4 = 1$, so that

$$p_1 D^+ F(r) + p_2 D_+ F(r) + p_3 D^- F(r) + p_4 D_- F(r) = 0. \tag{9}$$

Multiplying (7) and (8) respectively by p_k $(k = 1, 2, 3, 4)$ and adding, using (9), the Theorem follows.

3. Theorem 4 can be used to prove the following theorem which generalizes the Lemma proved in [7].

Theorem 5. *Let the following conditions be fulfilled for functions* f *and* g:

1) $f(x),\ g(x) \in C_{[a,\, b]}$,

2) *there exists* D^+, D_+, D^-, D_- *in* (a, b), *so that*

$$q_1 D^+ f(x) + q_2 D_+ f(x) + q_3 D^- f(x) + q_4 D_- f(x) \neq 0$$

for all $x \in (a, b)$ *and all numbers* $q_k \geqq 0$ $(k = 1, 2, 3, 4)$ *with* $q_1 + q_2 + q_3 + q_4 = 1$,

3) $g'_+(x)$ *and* $g'_-(x)$ *exists for* $x \in (a, b)$ *and* $r_1 g'_+(x) + r_2 g'_-(x) \neq 0$ *for all* $x \in (a, b)$ *and all* $r_k \geqq 0$ $(k = 1, 2)$ *with* $r_1 + r_2 = 1$,

4) $g(a) \neq g(b)$.

Then for all natural numbers m, *there exist different points* $x_k \in (a, b)$ $(k = 1, \ldots, m)$ *so that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{1}{m} \sum_{k=1}^{m} \frac{p_1^{(k)} D^+ f(x_k) + p_2^{(k)} D_+ f(x_k) + p_3^{(k)} D^- f(x_k) + p_4^{(k)} D_- f(x_k)}{(p_1^{(k)} + p_2^{(k)})\, g'_+(x_k) + (p_3^{(k)} + p_4^{(k)})\, g'_-(x_k)}$$

where $p_i^{(k)} \geqq 0$ $(k = 1, \ldots, m;\ i = 1, 2, 3, 4)$ *are some real numbers with* $p_1^{(k)} + p_2^{(k)} + p_3^{(k)} + p_4^{(k)} = 1$ *for* $k = 1, \ldots, m$.

Proof. In the prove of the generalized theorem in [7] one uses the ordinary CAUCHY's theorem. The proof of this theorem is the same, except that CAUCHY's theorem is replaced by the Theorem 4.

From Theorem 5 we obtain:

Theorem 6. *If the conditions* 1, 2, 3, *from Theorem* 5 *are fulfilled for a function* f, *then for every natural number* m *there exists real numbers* $x_k \in (a, b)$ $(k = 1, \ldots, m)$ *so that*

$$f(b) - f(a) = \frac{1}{m} \sum_{k=1}^{m} (p_1^{(k)} D^+ f(x_k) + p_2^{(k)} D_+ f(x_k) + p_3^{(k)} D^- f(x_k) + p_4^{(k)} D_- f(x_k))$$

where $p_i^{(k)} \geqq 0$ $(k = 1, \ldots, m;\ i = 1, 2, 3, 4)$ *and* $p_1^{(k)} + p_2^{(k)} + p_3^{(k)} + p_4^{(k)} = 1$ *for all* $k = 1, \ldots, m$.

Proof. In Theorem 5 put $g(x) = x$. g satisfies the conditions of the Theorem 5. This comletes the proof.

A consequence of Theorem 6 is:

Theorem 7. *If the conditions of Theorem* 6 *are fulfilled for a function* f *and if* $f(a) = f(b)$, *then for every natural number* m, *there exist real numbers* $x_k \in (a, b)$ *so that*

$$\sum_{k=1}^{m} (p_1^{(k)} D^+ f(x_k) + p_2^{(k)} D_+ f(x_k) + p_3^{(k)} D^- f(x_k) + p_4^{(k)} D_- f(x_k)) = 0$$

where $p_i^{(k)} \geqq 0$ $(k = 1, \ldots, m;\ i = 1, 2, 3, 4)$ *and* $p_1^{(k)} + p_2^{(k)} + p_3^{(k)} + p_4^{(k)} = 1$ *for all* $k = 1, \ldots, m$.

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