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346. ON A FUNCTIONAL DIFFERENTIAL EQUATION*

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Consider the equation with respect to the unknown function $y(x)$

(1) $$y'(x) = A(x)y(f(x)) + B(x)$$

where

(2) function f maps an open set D onto D.

Function f can be iterated in the following way

(3) $$f_1(x) = f(x)\ , \ldots , \ f_k(x) = f(f_{k-1}(x))\ , \ldots , \ f_n(x) = x \ \ (x \in D)$$

where n is the least natural number for which the last relation (3) holds.

(4) Functions A, B and f are $n-1$ times differentiable on D, and y is n times differentiable on the same set.

The following example shows that there exists such a function f.

$$f(x) = \begin{cases} x, & x \in (-\infty, 0) \cup (n, +\infty), \\ x+1, & x \in (0, 1) \cup (1, 2) \cup \cdots \cup (n-2, n-1), \\ x-(n-1), & x \in (n-1, n) \end{cases}$$

where $D = (-\infty, 0) \cup (0, 1) \cup \cdots \cup (n-1, n) \cup (n, +\infty)$.

Introduce the notation $g(f_k) = g(f_k(x))$ for any function g.

In this paper we shal prove the following theorem:

Theorem. *Equation* (1), *for which conditions* (2)—(4) *hold, can be reduced a linear differential equation of order n.*

Proof. Differentiating (1) $n-1$ times, we obtain

(5) $$\begin{aligned} y''(x) &= A'y(f) + Ay'(f)f' + B', \\ y'''(x) &= A''y(f) + 2A'y'(f')f + ay''(f)f'^2 + Ay'(f)f'' + B'', \\ &\vdots \\ y^{(n-1)}(x) &= g_{n-1}(x, y(f), \ y'(f), \ldots , y^{(n-2)}(f)), \\ y^{(n)}(x) &= g_n(x, y(f), \ y'(f), \ldots , y^{(n-2)}(f), \ y^{(n-1)}(f)). \end{aligned}$$

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Using (1) and the first n—2 equations of system (5), by GAUSS' method we find

(6) $$y^{(k)}(f)=R_k(x,\ y'(x),\ \dots,\ y^{(k+1)}(x))\quad (k=0,\ 1,\ \dots,\ n-2).$$

In virtue of (6), the last equation of (5) becomes

(7) $$y^{(n)}(x)=H_{n-1}(x,\ y'(x),\ \dots,\ y^{(n-1)}(x),\ y^{(n-1)}(f)).$$

Replacing x by $f(x)$ in the last but one equation of (5) we get

(8) $$y^{(n-1)}(f)=g_{n-1}(f,\ y(f_2),\ y'(f_2),\ \dots,\ y^{(n-3)}(f_2),\ y^{(n-2)}(f_2)).$$

Replacing x by $f(x)$ in (6) for $0\leqq k\leqq n-3$, and substituting the obtain results in (8) and (7), in virtue of (6), we have

$$y^{(n)}(x)=H_{n-2}(x,\ y'(x),\ \dots,\ y^{(n-1)}(x),\ y^{(n-2)}(f_2)).$$

Continuing this procedure, we obtain

(9) $$y^{(n)}(x)=H_{n-r}(x,\ y'(x),\ \dots,\ y^{(n-1)}(x),\ y^{(n-r)}(f_r))\quad (r<n).$$

According to (1) and (2), (9) yields for $r=n-1$

(10) $$y^{(n)}(x)=H_1(x,\ y'(x),\ \dots,\ y^{(n-1)}(x),\ A(f_{n-1})\,y(x)+B(f_{n-1})).$$

Since equations (1) and (5) are linear we conclude that (10) is a linear differential equation of order n. Let the solution of equation (10) be given by

(11) $$y=h(x,\ C_1,\ C_2,\ \dots,\ C_n).$$

If there is a (n times differentiable) solution of (1), it has to have the form (11). Substituting (11) into (1) we obtain the solution of equation (1). This comletes the proof of the theorem.

EXAMPLE. Consider the equation $y'(x)=y(f)$ where $f(x)=(1-x)^{-1}$ and $D=(-\infty,0)\cup(0,1)\cup(1,+\infty)$. For f we have $f_3(x)=x$ for $x\in D$. In this case equation (10) reads

$$x^2(1-x)^2y'''(x)-2x^2(1-x)y''(x)-y(x)=0.$$

Remark. Equation (1) was considered in [1] for $f(x)=\dfrac{1}{x}$, and for some special cases of A and B.

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