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344. EXPANSION OF POWERS OF A CLASS OF LINEAR DIFFERENTIAL OPERATORS*

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In two previous papers [1, 2], we had shown that

$$x^n D^{2n} \equiv [xD^2-(n-1)\,D]^n, \quad x^{2n} D^n \equiv [x^2 D-(n-1)\,x]^n \tag{1}$$

plus generalizations for the operators

$$x^{rn} D^{(r+1)n} \quad \text{and} \quad x^{(r+1)n} D^{rn}.$$

These identities were then used to solve certain n-th order linear differential equations. In a subsequent paper, BERKOVIČ and KVALWASSER [3] generalized the identities (1) to

$$[xD^2+aD]^n \equiv \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(a+n)}{\Gamma(a+n-k)} x^{n-k} D^{2n-k}, \tag{2}$$

$$[x^2 D+ax]^n \equiv \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(a+n)}{\Gamma(a+n-k)} x^{2n-k} D^{n-k} \tag{3}$$

and used these to solve certain other n-th order linear differential equations. Here we obtain a still further generalization by finding, in a more direct manner, the polynomial expansions of

$$[(xD+a+1-n)\,(xD+a+1-2n)\cdots(xD+a+1-rn)\,D]^n, \tag{4}$$

$$[(x\,(xD+a+1-n)\,(xD+a+1-2n)\cdots(xD+a+1-rn)]^n. \tag{5}$$

We now give a detailed derivation of (4) and (5) for the case $r=1$ (corresponding to (2) and (3)) and then sketch the extensions for general r.

In

$$[x^2 D+ax]^n \equiv x^{2n} D^n + A_1 x^{2n-1} D^{n-1} + \cdots + A_n x^n,$$

let $x=e^z$ to give

$$[e^z\,(D+a)]^n \equiv e^{nz}\,[A_n + A_{n-1} D + \cdots + D\,(D-1)\cdots(D-n+1)].$$

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Using the exponential shift theorem of the l.h.s., we obtain

$$(D+a)(D+a+1)\cdots(D+a+n)\equiv A_n+A_{n-1}D+\cdots+D(D-1)\cdots(D-n+1).$$

Setting $D=0, 1, 2, \ldots$, successively, we obtain the following set of triangular linear equations:

$$A_n=\Gamma(a+n)/\Gamma(a),$$
$$A_n+A_{n-1}=\Gamma(1+a+n)/\Gamma(1+a),$$
$$A_n+2A_{n-1}+2A_{n-2}=\Gamma(2+a+n)/\Gamma(2+a), \text{ etc.}$$

Solving successively, we get

$$A_{n-1}=\frac{n}{a}\frac{A_n}{1!},\quad A_{n-2}=\frac{n(n-1)}{a(a+1)}\frac{A_n}{2!},\ldots,$$

Then by guessing,

$$A_{n-s}=\frac{n(n-1)\cdots(n-s+1)}{a(a+1)\cdots(a+s-1)}\frac{A_n}{s!}.$$

To verify our guess, we have to show that

$$A_n+rA_{n-1}+r(r-1)A_{n-2}+\cdots+r!\,A_{n-r}=\Gamma(r+a+n)/\Gamma(r+a)$$

or that

$$\sum_{s=0}^{r}\frac{(-r)_s(-n)_s}{s!\,(a)_s}=\frac{\Gamma(r+a+n)}{\Gamma(r+a)}\frac{\Gamma(a)}{\Gamma(a+n)}.$$

The latter follows immediately from GAUSS' theorem [4]

$$F(a, b; c; 1)=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\,(\mathrm{Re}(c-a-b)>0)$$

where as usual

$$(a)_s=a(a+1)\cdots(a+s-1);\qquad (a)_0=1,$$
$$F(a, b; c; z)=1+\frac{ab}{1!\,c}z+\frac{a(a+1)\,b(b+1)}{2!\,c(c+1)}z^2+\cdots.$$

Since

$$A_r=\binom{n}{r}\frac{\Gamma(a+n)}{\Gamma(a+n-r)},$$

we have derived identity (3) of BERKOVIČ and KVALWASSER.

Identity (2) will follow more simply from (3). In

$$[xD^2+aD]^n\equiv x^n D^{2n}+B_1x^{n-1}D^{2n-1}+\cdots+B_n D^n$$

let $x=e^z$ to give

$$\{e^{-z}[D^2+(a-1)D]\}^n\equiv D(D-1)\cdots(D-2n+1)+B_1D(D-1)\cdots(D-2n+2)$$
$$+\cdots+B_nD(D-1)\cdots(D-n+1).$$

Using the exponential shift theorem of the l.h.s., we obtain

$$(D+a-1)(D+a-2)\cdots(D+a-n)$$
$$\equiv B_n+B_{n-1}(D-n)+B_{n-2}(D-n)(D-n-1)+\cdots.$$

On letting $D-n=D'$, we obtain the same set of equations as for the A_r's. Thus $A_r=B_r$ which establishes (2).

To establish the expansion of (5), we proceed in the same way. In

$$[x(xD+a+1-n)(xD+a+1-2n)\cdots(xD+a+1-rn)]^n$$
$$\equiv x^{(r+1)n}D^{rn}+C_1x^{(r+1)n-1}D^{rn-1}+\cdots,$$

we again let $x=e^z$ and use the exponential shift theorem to give

$$(D+a)(D+a-1)\cdots(D+a-rn+1)\equiv C_{rn}+C_{rn-1}D+C_{rn-2}D(D-1)+\cdots.$$

Solving, we find that

$$C_{rn}=\frac{\Gamma(a+1)}{\Gamma(a+1-rn)},$$

$$C_{rn-s}=\frac{(-1)^sC_{rn}(-rn)_s}{s!(a+1-rn)_s}.$$

Whence,

(6) $$[x(xD+a+1-n)(xD+a+1-2n)\cdots(xD+a+1-rn)]^n$$
$$\equiv\sum_{k=0}^{rn}\binom{rn}{k}\frac{\Gamma(a+1)}{\Gamma(a+1-k)}x^{(r+1)n-k}D^{rn-k}$$

where $r=1,2,3,\ldots$. Proceeding in the same way for (4), we find the coefficients satisfy the same set of linear equations as (5). Thus,

(7) $$[(xD+a+1-n)(xD+a+1-2n)\cdots(xD+a+1-rn)D]^n$$
$$\equiv\sum_{k=0}^{rn}\binom{rn}{k}\frac{\Gamma(a+1)}{\Gamma(a+1-k)}x^{rn-k}D^{(r+1)n-k}.$$

By letting $r=1$, $a+1-n=a'$, (6) and (7) reduce to (3) and (2).

Since (2) and (3) remain valid if we replace x by $-D$ and D by x, we get the dual identities

(8) $$[Dx^2-ax]^n\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(a+n)}{\Gamma(a+n-k)}D^{n-k}x^{2n-k},$$

(9) $$[D^2x-aD]^n\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(a+n)}{\Gamma(a+n-k)}D^{2n-k}x^{n-k}.$$

Since

$$D^{n-k}x^{2n-k}\equiv\sum_{i=0}^{n-k}\binom{n-k}{i}(D^ix^{2n-k})D^{n-k-i}\equiv\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(2n-k)!}{(2n-k-i)!}x^{2n-k-i}D^{n-k-i},$$

(8) can be rewritten (after letting $k+i=j$ and interchanging the order of summations) as

(10) $$[x^2D+(2-a)x]^n\equiv\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^k\binom{n}{k}\binom{2n-k}{j-k}\frac{(n-k)!}{(n-j)!}\frac{\Gamma(a+k)}{\Gamma(a+n-k)}x^{n-j}D^{2n-j}.$$

Comparing (10) with (3), we obtain the combinatorial indentity

$$(11)\qquad \frac{\Gamma(2-a+n)(2n-j)!}{\Gamma(2-a+n-j)\Gamma(a+n)} \equiv \sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{(2n-k)!}{\Gamma(a+n-k)}, \quad 0 \leqq j \leqq n.$$

Expanding out (9) in a similar fashion also leads to (11).

Using (6), we can now solve the $(2n)$-th order differential equation

$$\sum_{i=0}^{2n}\binom{2n}{i}\frac{\Gamma(a+1)}{\Gamma(a+1-i)}x^{3n-i}D^{2n-i}y = b^n y.$$

The solution will be a linear combination of the solutions of

$$(12)\qquad x(xD+a+1-n)(xD+a+1-2n)y = \lambda y$$

where

$$\lambda/b = 1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1}$$

and ω is a primitive n-th root of unity.

(12) can be rewritten as

$$\{x^2D^2 + [3(1-n)+2a]xD + (a+1-n)(a+1-2n) - \lambda/x\}y = 0.$$

This is a modified Bessel equation whose solution is given by [4]

$$y = x^{(1+\alpha)/2}Z_\nu(-2\sqrt{-\lambda/x})$$

where

$$\alpha = 3(1-n)+2a,$$

$$c = (a+1-n)(a+1-2n),$$

$$\nu = -\{(1-\alpha)^2 - 4c\}^{1/2}.$$

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