PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 338 — № 352 (1971)

343. AN INEQUALITY FOR THE SUM OF UNIT VECTORS*

Murray S. Klamkin and *Donald J. Newman*

It is shown that if $V_1, \ldots, V_r$ denote unit vectors in E^n which do not lie in any half-space, then $|\sum V_i| < r-2$.

In a short note [1], D. S. Mitrinović reviewed the history of some inequalities giving lower bounds for the resultant of a sum of vectors. Here, we give a corresponding inequality for an upper bound when the vectors are of unit length. This inequality is a generalization of the plane case due to A. J. Goldstein (Bell Telephone Laboratories), i.e., if P is a point inside a convex n-gon and if $V_1, \ldots, V_n$ denote unit vectors from P to the vertices of the polygon, then $|\sum V_i| < n-2$. The result is best possible, since one can get arbitrarily close to $n-2$ by considering a sequence of polygons converging to the degenerate case of a segment. We prove more generally that

Theorem. *If $V_1, \ldots, V_r$ donote unit vectors in E^n which do not lie in any half-space, then $|\sum V_i| < r-2$.*

Again the result is best possible since one can get arbitrarily close to $r-2$ by considering a sequence of convex polytopes converging to the degenerate case of a segment.

Proof. Since the V_i's do not all lie in any half-space, the convex cone which is positively spanned by the vectors must be E^n. For if it was not, there would then exist some hyperplane [2] such that the cone would lie entirely to one side of it. But this would violate the hypothesis. Consequently, there exists a set of numbres $a_1, \ldots, a_r$ where $a_i \geqq 0$, $i = 1, \ldots, r$, $\sum a_i = 1$ such that

$$a_1 V_1 + \cdots + a_r V_r = 0.$$

We now show that $a_i \leqq 1/2$. Assume the contrary that $a_1 > 1/2$. Then

$$|a_2 V_2 + \cdots + a_r V_r| = |-a_1 V_1|.$$

* Presented December 15, 1970 by D. S. Mitrinović.

The r.h.s. $>1/2$ and the l.h.s. $\leqq a_2+\cdots+a_r=1-a_1<1/2$ which is a contradiction. Finally, since

$$V_1+\cdots+V_r=(1-2a_1)V_1+\cdots+(1-2a_r)V_r,$$
$$|V_1+\cdots+V_r|<(1-2a_1)+\cdots+(1-2a_r)=r-2.$$

REFERENCES

1. D. S. Mitrinović: *An Old Inequality Rediscovered by Wilf.* These Publications № **181**—№ **196** (1967), 39—40.
2. C. Davis: *Theory of Positive Linear Dependence.* Amer. J. Math. **76** (1954), 733—746.

Theoretical Sciences Department
Scientific Research Staff,
Ford Motor Company,
Dearborn, Michigan 48121

Yeshiva University,
New York City, N. Y.

Note added in proof. The stated theorem can be extended for the case of non-unit vectors, i.e.,

Theorem. *If $V_1, \ldots, V_r$ denote vectors in E^n which do not lie in any half-space, then*

$$\left|\sum V_i\right| \leqq \sum |V_i| - 2\min_i |V_i|.$$