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342. GEOMETRIC INEQUALITIES AND THEIR GEOMETRY*

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1. Introduction. Recently a team of five authors[1] published a collection of over 400 geometric inequalities, most of them dealing with triangles. The majority of the latter can be rewritten in the form $P(a, b, c)>0$ or $P(a, b, c)\geqq 0$ where $P(a, b, c)$ is a symmetric and homogeneous polynomial in the real variables a, b, c, representing the sides of a triangle. In GI a great number of discrete polynomials $P(a, b, c)$ is given. In this paper we determine the complete set of symmetric and homogeneous polynomials of order $n\leqq 3$ that give rise to a correct geometric inequality and give some partial results for $n=4$.

2. Preliminary remarks. If $P(a, b, c)>0$ or $P(a, b, c)\geqq 0$ is a geometric inequality and if P is symmetric and homogeneous we will call it an inequality polynomial or I.P. Many I.P.'s published in GI have the special property that they vanish identically for equilateral triangles. In such a case P will be called a special I.P. Now the symmetric and homogeneous polynomials of order n form a vector space V_n of finite dimension[2]. If P_1 and P_2 are I.P.'s then also $\lambda_1 P_1+\lambda_2 P_2$ is one when λ_1 and λ_2 are non-negative, not both zero. So these polynomials form a convex subset of V_n which is the inner part C_n of a semicone.

The polynomial

$$P_{pqr}=a^p b^q c^r+a^p b^r c^q+a^q b^p c^r+a^q b^r c^p+a^r b^p c^q+a^r b^q c^p$$

where $p\geqq q\geqq r$ is supposed, is a symmetric and homogeneous polynomial of order $n=p+q+r$. Any symmetric and homogeneous polynomial of order n can be written as a linear combination $\sum \lambda_{pqr} P_{pqr}$ of such polynomials. Each of these polynomials takes the value 6 in the point (1, 1, 1). So the special I.P's all lie in a hyperplane H_n with equation $\sum \lambda_{pqr}=0$. The set of special I.P.'s is a convex and semiconic subset C_n^* of this hyperplane; we have $C_n^*=C_n\cap H_n$.

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* Presented October 1, 1970 by O. BOTTEMA.

[1] O. BOTTEMA, R. Ž. ĐORĐEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ, P. M. VASIĆ: *Geometric Inequalities*. Groningen 1969. It will be denoted GI in this paper.

[2] For $n=6k$ this dimension is $3k^2+3k+1$, for $n=6k+i$ it is $(k+1)(3k+i)$; $i=1, 2, 3, 4, 5$.

We order the polynomials P_{pqr} by writing their leading terms in alphabetic order. Then the polynomials P'_{pqr} obtained by subtracting its successor from each P_{pqr} but the last one form a basis of H_n.

If $\varrho>0$ then $P(a, b, c)$ and $P(\varrho a, \varrho b, \varrho c)$ have equal signs because P is homogeneous. Therefore we need only to consider classes of similar triples. The classes of similar triples with positive elements form the inner part of the triangle (1, 0, 0), (0, 1, 0), (0, 0, 1) in the projective plane. The coordinates a, b, c have to satisfy $a>b+c$, $b>c+a$, $c>a+b$. This reduces the part of the plane to be considered to the inner part of the triangle (1, 1, 0), (0, 1, 1), (1, 0, 1).

Because P is symmetric, a permutation of a, b, c does not change its value. Hence without loss of generality we may assume $a \geqq b \geqq c$. This reduces the part of the plane to be considered to the inner part of the triangle Δ: (1, 1, 1), (1, 1, 0), (2, 1, 1). Occasionally we will choose $b=1$, $a=1+\alpha$, $c=1-\gamma$, and study the values of P on the euclidean triangle T: $\alpha, \gamma \geqq 0$, $\alpha+\gamma<1$. This will not lead to confusion because points of Δ are denoted with three and points of T with two coordinates.

3. I.P.'s of order 1. Here $X'_1=\frac{1}{2}P_{100}=a+b+c=2s$ is a basis of V_1. The I.P. semicone is the set $x_1X'_1$ with $x_1>0$. The set of special I.P.'s of order 1 is empty.

4. I.P.'s of order 2. $P_{200}=2a^2+2b^2+2c^2$ and $P_{110}=2ab+2bc+2ca$ form : basis of V_2, while $P'_{200}=Q=(a-b)^2+(b-c)^2+(c-a)^2$ is a basis of H_2. W write $X_1^2=P'_{200}$.

Then the semicone of special I.P.'s is the set $x_1X_1^2$ with $x_1>0$. Anoth basis of V_2 is given by X_1^2 and

$$X_2^2=\frac{1}{2}(P_{110}-P'_{200})=2ab+2bc+2ca-a^2-b^2-c^2.$$

Here also X_2^2 is an I.P. for $X_2^2=a^2-(b-c)^2+b^2-(c-a)^2+c^2-(a-b)^2>0$. the I.P. semicone contains the set $S: x_1X_1^2+x_2X_2^2$; $x_1 \geqq 0$, $x_2 \geqq 0$; not $x_1=x_2=$ On the other hand, if $P=x_1X_1^2+x_2X_2^2$, then $P(1, 1, 0)=2x_1$ and $P(1, 1, 1)=3$ so if P is an I.P. then both x_1 and x_2 are nonnegative.

So the I.P. semicone is exactly the set S.

5. I.P.'s of order 3. A basis for H_3 is given by

$$X_1^3=P'_{300}-P'_{210}=(a-b)^2(a+b-c)+(b-c)^2(b+c-a)+(c-a)^2(c+a-l$$

and

$$X_2^3=3P'_{210}-P'_{300}=(a-b)^2(3c-a-b)+(b-c)^2(3a-b-c)+(c-a)^2(3b-a$$

Evidently, X_1^3 is an I.P. Also X_2^3 is one; we have

$$X_2^3=(a-b)^2(3c-a-b)+(b-c)^2(3a-b-c)+\{(a-b)+(b-c)\}^2(3b-a-$$
$$=2(a-b)^2(b+c-a)+2(b-c)^2(a+b-c)+2(a-b)(b-c)(3b-a-c)$$

because

$$3b-a-c>(3b-c)-(b+c)=2(b-c).$$

Further, if $P=x_1X_1^3+x_2X_2^3$ then $P(2, 1, 1)=4x_1$ and $P(1, 1, 0)=4x_2$. So in an I.P. neither x_1 nor x_2 can be negative.

The semicone of special I.P.'s therefore is the set $x_1X_1^3+x_2X_2^3$, $x_1 \geqq 0$, $x_2 \geqq 0$, not $x_1=x_2=0$.

As for the other I.P.'s certainly $X_3^3=(a+b-c)(b+c-a)(c+a-b)$ is one. Consider the set

$$\{P \mid P=x_1X_1^3+x_2X_2^3+x_3X_3^3\}.$$

We have $P(2, 1, 1)=4x_1$; $P(1, 1, 0)=4x_2$; $P(1, 1, 1)=x_3$. So the I.P. semicone is the above set under the condition $x_1, x_2, x_3 \geqq 0$, not $x_1=x_2=x_3=0$.

6. I.P.'s of order 4. $P_{400}, P_{310}, P_{220}, P_{211}$ form a basis of V_4; $P'_{400}, P'_{310}, P'_{220}$ form a basis of H_4. Another basis of the latter space is given by

$$X_1^4=\frac{1}{2}(P'_{400}-P'_{310}-P'_{220})$$

$$=\frac{1}{2}\{(a-b)^2(a^2+b^2-c^2)+(b-c)^2(b^2+c^2-a^2)+(c-a)^2(c^2+a^2-b^2)\}$$

$$=a^2(a-b)^2+c^2(b-c)^2+(a-b)(b-c)(c^2+a^2-b^2),$$

$$X_2^4=\frac{1}{2}(P'_{400}-5P'_{310}+3P'_{220})$$

$$=\frac{1}{2}\{(a-b)^2(a^2+b^2+3c^2-4ab)+(b-c)^2(b^2+c^2+3a^2-4bc)$$
$$+(c-a)^2(c^2+a^2+3b^2-4ac)\}$$

$$=\frac{1}{2}(a-b)^2\{(a-2b)^2+(a-2c)^2\}+\frac{1}{2}(b-c)^2\{(2a-c)^2+(2b-c)^2\}$$
$$+(a-b)(b-c)(c^2+a^2+3b^2-4ac)$$

$$X_3^4=\frac{1}{2}(-P'_{400}+3P'_{410}+P'_{220})$$

$$=\frac{1}{2}\{(a-b)^2(c^2-(a-b)^2)+(b-c)^2(a^2-(b-c)^2)+(c-a)^2(b^2-(c-a)^2)$$

Evidently X_1^4 and X_3^4 are I.P.'s. X_2^4 is another one because

$$c^2+a^2+3b^2-4ac=(a-2c)^2+3(b^2-c^2).$$

Now, let

$$P=x_1X_1^4+x_2X_2^4+x_3X_3^4.$$

We have $P(2, 1, 1)=4x_1$; $P(1, 1, 0)=4x_2$; so in an I.P. both x_1 and x_2 l to be nonnegative.

For x_1, x_2 positive, x_3 negative, $x_1-x_2 \geqq 0$ we have

$$(x_1+x_2-x_3)P(0, y)=[y^2(x_1+x_2-x_3)-y(x_1-x_2)]^2+y^2(4x_1x_2-x_3{}^2).$$

Hence

$$P\left(0, \frac{x_1-x_2}{x_1+x_2-x_3}\right)=\frac{(4x_1x_2-x_3^2)(x_1-x_2)^2}{(x_1+x_2-x_3)^3},$$

and, since $0 \leqq \frac{x_1-x_2}{x_1+x_2-x_3} < 1$, for an I.P. $x_3^2 \leqq 4x_1x_2$ is required.

For x_1, x_2 positive, x_3 negative, $x_2-x_1 \geqq 0$, we have

$$(x_1+x_2-x_3)P(\alpha, 0)=[\alpha^2(x_1+x_2-x_3)-\alpha(x_2-x_1)]^2+\alpha^2(4x_1x_2-x_3^2).$$

Apparently also here P can be an I.P. but if $x_3^2 \leqq 4x_1x_2$.

Now consider the points in H_4 for which $x_3^2=4x_1x_2$, $x_3<0$; i.e., consider the polynomials $P=t^2X_1+X_2-2tX_3$, $t>0$. Then

$$(\alpha^2+\alpha\gamma+\gamma^2)P=[(\alpha^2+\alpha\gamma+\gamma^2)(t-1)+(\alpha^3-\gamma^3)(t+1)+(\alpha^2\gamma-\alpha\gamma^2)(t+2)]^2$$
$$+3\alpha^2\gamma^2(\alpha+\gamma)^2(t+2)^2 \geqq 0,$$

with equality only for $\alpha=0$, $\gamma=0$, for any t; for $\alpha=0$, $\gamma=\frac{t-1}{t+1}$ if $t>1$; for $\gamma=0$, $\alpha=\frac{t-1}{t+1}$ if $t<1$.[1] So in H_4 the semicone C_4^* of special I.P.'s is bounded by the cone $x_3^2=4x_1x_2$ and the tangent planes $x_1=0$, $x_2=0$.

In V_4 we can take as a basis the set $\{X_1^4, X_2^4, X_3^4, X_4^4\}$ where

$$X_4^4=F^2=s(s-a)(s-b)(s-c).$$

In an I.P. of the form $P=x_1X_1^4+x_2X_2^4+x_3X_3^4+x_4X_4^4$ we must have $x_1 \geqq 0$, $x_2 \geqq 0$, $x_4 \geqq 0$, since $P(2, 1, 1)=4x_1$; $P(1, 1, 0)=4x_2$; $P(1, 1, 1)=\frac{3x_4}{16}$. For fixed $x_1, x_2, x_3>0$ we have to find the minimal value of x_3 for which P is still nonnegative definite on Δ or T. In that case there is a point Q of $\bar{\Delta}$ for which P vanishes. If this point is an inner point we must have

$$\sum x_i\frac{\partial X_i}{\partial a}=\sum x_i\frac{\partial X_i}{\partial b}=\sum x_i\frac{\partial X_i}{\partial c}=0.$$

We obtain 3 homogeneous linear equations in x_1, x_2, x_3, x_4 which are independent because in an inner point of Δ we have $a \neq b \neq c \neq a$.

We will not carry out this computation here but we give the result in the form of the following

Theorem. *If $\Delta_0=(a_0, b_0, c_0)$ is any triangle, then the polynomial*

$$\varphi(a, b, c)=2(a_0^2+b_0^2+c_0^2)(a_0+b_0+c_0)^2(ab+bc+ca)^2$$
$$+(a_0b_0+b_0c_0+c_0a_0)(a_0+b_0+c_0)^2(a^2+b^2+c^2)^2$$
$$-(a_0^2+b_0^2+c_0^2)(a_0b_0+b_0c_0+c_0a_0)(a+b+c)^4$$

[1] So for all positive t the semidefinite form vanishes for the class of equilateral triangles and in addition to that for exactly one class of similar isosceles triangles; conversely, for each class of similar isosceles triangles it is possible to construct a special I.P. that vanishes just for that class.

is an I.P. vanishing for all points inside Δ *lying on the conic*

$$(a_0^2+b_0^2+c_0^2)(ab+bc+ca)=(a_0b_0+b_0c_0+c_0a_0)(a^2+b^2+c^2).$$

It passes through Δ_0.

Proof. Without loss of generality we may assume $a+b+c=a_0+b_0+c_0$. Let $a_0^2+b_0^2+c_0^2=u$, $a_0b_0+b_0c_0+c_0a_0=v$, $ab+bc+ca=v+w$, then

$$a^2+b^2+c^2=u-2w.$$

We have

$$\varphi=(a_0+b_0+c_0)^2\{2u(v+w)^2+v(u-2w)^2-uv(u+2v)\}$$
$$=2(a_0+b_0+c_0)^4w^2\geqq 0.$$

If in φ we determine the coefficients x_1, x_2, x_3, x_4 we obtain

$$x_1=(5u-6v)^2,\quad x_2=(u-2v)^2,\quad x_3=(u-2v)(14u-12v),\quad x_4=48(u-v)^2.$$

Indeed $x_1\geqq 0$, $x_2\geqq 0$, $x_4\geqq 0$, $x_3<0$ since $v\leqq u<2v$.

Since the two proportions $x_1:x_2:x_4$ depend on one parameter u/v only the I.P.'s of this type exist for special triples (x_1, x_2, x_4) only. Indeed we have

$$48uv=-18x_1+66x_2+8x_4;\quad 48v^2=-12x_1+60x_2+5x_4;$$
$$48u^2=-24x_1+72x_2+12x_4;$$

so x_1, x_2, x_4 must satisfy the quadratic relation

$$(-18x_1+66x_2+8x_4)^2=(-24x_1+72x_2+12x_4)(-12x_1+60x_2+5x_4),$$

or

$$9x_1^2-18x_1x_2+9x_2^2-6x_1x_4-6x_2x_4+x_4^2=0.$$

In the space spanned by X_1^4, X_2^4, X_4^4 this represents a cone inscribed in the trihedral angle bounded by $x_1=0$, $x_2=0$, $x_4=0$.

For all other vectors (x_1, x_2, x_4) the corresponding I.P. vanishes in a boundary point of $\bar{\Delta}$ which represents an isosceles triangle with vertical angle $<\frac{\pi}{3}$ (if on the segment between (1, 1, 0) and (1, 1, 1)); an isosceles triangle with vertical angle $>\frac{\pi}{3}$ (if on the segment between (1, 1, 1) and (2, 1, 1)); or a degenerate triangle (if on the segment between (2, 1, 1) and (1, 1, 0)).

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