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338. ON A FUNCTIONAL EQUATION*

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D. S. Mitrinović and S. B. Prešić (see: [1]) have proved that the general solution of the functional equation

$$(1) \qquad f(x_1, x_2)f(x_3, x_4)+f(x_1, x_3)f(x_4, x_2)+f(x_1, x_4)f(x_2, x_3)=0,$$

where $f\colon R\times R\rightarrow R$ is an unknown function, is given by

$$(2) \qquad f(x, y)=\begin{vmatrix} F(x) & G(x) \\ F(y) & G(y) \end{vmatrix} \qquad (F, G\colon R\rightarrow R \text{ are arbitrary functions}).$$

After the above mentioned papers the whole series of papers appeared wherein various generalizations of functional equation (1) are given. However, these generalizations tended to define a functional equation, the solution of which would be a function represented in the form of the determinant analog to (2).

Natural extension of the equation (1) for the case $f\colon R\times R\times R\rightarrow R$ would be

$$(3) \quad f(x_1, x_2, x_3)f(x_4, x_5, x_6)+f(x_1, x_2, x_4)f(x_5, x_6, x_3)$$
$$+f(x_1, x_2, x_5)f(x_6, x_3, x_4)+f(x_1, x_2, x_6)f(x_3, x_4, x_5)=0$$

(the equation (1) is cyclic according to x_2, x_3, x_4, while the equation (3) is cyclic according to x_3, x_4, x_5, x_6). This equation is the subjet of the present paper.

If we exclude the trivial solution $f(x, y, z)=0$, then there are three real numbers a, b, c such that $f(a, b, c)\neq 0$. Putting $x_1=a$, $x_2=b$, $x_3=x_4=x_5=c$, $x_6=u$, we obtain

$$(4) \qquad f(c, c, u)+f(c, u, c)+f(u, c, c)=0,$$

since $f(x, x, x)\equiv 0$ for all x.

Equation (3) for $x_1=x_2=x_4=x_5=c$, $x_3=x_6=u$ becomes

$$f(c, c, u)\big(f(c, c, u)+f(u, c, c)\big)=0,$$

i.e., according to (4),

$$(5) \qquad f(c, c, u)f(c, u, c)=0.$$

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Let us mark by C_1 the class of all functions f, being the solutions of the equation (3) and for which $f(c, c, u)\not\equiv 0$, and by C_2 the class of all functions f being the solutions of the equation (3) and for which $f(c, c, u)\equiv 0$.

Class C_1. Since $f(c, c, u)\not\equiv 0$, there exists a number d $(d\in R)$ such that $f(c, c, d)\neq 0$. Then, form (5) we find $f(c, d, c)=0$ and from (4) we find $f(c, c, d)=-f(d, c, c)$.

If we put $x_1=x_2=x_4=x_6=c$, $x_3=u$, $x_5=d$, from (3) we obtain

$$f(c, u, c)=0,$$

Then (3) becomes

$$f(c, c, u)=-f(u, c, c).$$

If we set $x_1=x_5=x_6=c$, $x_2=u$, $x_3=v$, $x_4=d$ in (3), we get

$$f(c, u, v)=-F(v)\,G(u),$$

where

(6) $$G(u)=f(c, u, d),\ F(v)=-\frac{f(c, c, v)}{f(c, c, d)}.$$

Setting in (3) $x_1=x$, $x_2=x_5=x_6=c$, $x_3=y$, $x_4=d$, we find

(7) $$f(x, c, y)=F(x)\,H(y)-F(y)\,K(x),$$

with

(8) $$K(x)=f(x, c, d),\ H(y)=f(c, y, d)+f(y, d, c).$$

$x_1=d$, $x_2=x_3=x_5=c$, $x_4=x$, $x_6=y$ leads by (3) to

(9) $$f(x, c, y)+f(y, c, x)=0.$$

If in (9) we set $x=y=d$, then we obtain $f(d, c, d)=0$.

If we put $x_1=x_2=x_6=c$, $x_3=x_4=x_5=d$, then from (3) we obtain

$$f(c, d, d)+f(d, d, c)=0.$$

From (6) and (8) we have

$$F(d)=-1,\ K(d)=0,\ H(d)=0.$$

Putting $x=d$ in (7) we find

$$f(d, c, y)=-H(y).$$

If we write in (3): $x_1=x_2=x_6=c$. $x_3=d$, $x_4=v$, $x_5=w$ we obtain

$$f(v, w, c)=-F(w)\,H(v)+F(v)\,H(w)+F(w)\,G(v).$$

From (7) and (9) follows

(10) $$-F(y)\,K(x)+F(x)\,H(y)-F(x)\,K(y)+F(y)\,H(x)=0.$$

For $y=d$, it follows from (10) that $K(x)=H(x)$ and (7) becomes

$$f(x, c, y)=F(x)\,H(y)-F(y)\,H(x).$$

Finally, if we write $x_1=d,\ x_2=x_3=c,\ x_4=u,\ x_5=v,\ x_6=w$ in (3), we find

$$(11)\qquad f(u,v,w)=\frac{-1}{f(c,c,d)}\big(H(u)H(w)F(v)+H(u)F(w)G(v)-H(v)F(u)H(w)-H(w)F(v)G(u)\big).$$

If we put $x_1=x_2=x_4=c,\ x_3=x_6=d,\ x_5=v$, then from (3) we obtain

$$\lambda F(v)=G(v)-H(v)\qquad (\text{with } \lambda=f(d,d,c)).$$

If $\lambda=0$, then $G(v)=H(v)$ and (11) becomes

$$(12)\qquad f(u,v,w)=F_2(v)\big(F_2(u)F_1(w)-F_2(w)F_1(u)\big)$$

where $F_2(v)=H(v)$ and $F_1(u)=\dfrac{F(u)}{f(d,c,c)}$.

If $\lambda\neq 0$, then $F(u)=\dfrac{1}{\lambda}\big(G(u)-H(u)\big)$ and (11) leads to (12) with

$$F_2(u)=G(u),\ F_1(u)=-\frac{H(u)}{f(d,c,c)f(d,d,c)}.$$

Conversely, the function (12) satisfies equation (3).

Class C_2. This class will be subdivided into two sub-classes: C_{21} the class of all functions f being the solution of the equation (3) for which $f(c,c,u)\equiv 0$ and $f(u,c,c)\not\equiv 0$ and C_{22} the class of all functions f being the solution of the equation (3) for which $f(c,c,u)\equiv 0$ and $f(u,c,c)\equiv 0$.

Class C_{21}. Since $f(u,c,c)\not\equiv 0$, there exists a number $e\,(e\in R)$ such that $f(e,c,c)\neq 0$. Putting in (3) $x_1=x,\ x_3=y,\ x_2=x_5=x_6=c,\ x_4=e$, we obtain

$$f(x,c,y)=F(x)G(y),$$

with

$$F(x)=-\frac{f(x,c,c)}{f(e,c,c)},\ G(y)=f(c,y,e)+f(y,e,c).$$

Similarly, for $x_1=x_5=x_6=c,\ x_2=u,\ x_3=v,\ x_4=e$ the equation (3) becomes

$$f(c,u,v)=-F(u)G(v).$$

If $x_1=e,\ x_2=x_5=x_6=c,\ x_3=v,\ x_4=e$, the equation (3) becomes

$$f(e,c,v)=-G(v).$$

If we now replace $x_1=e,\ x_2=x_3=x_6=c,\ x_4=u,\ x_5=v$, from (3) we obtain

$$f(u,v,c)=F(u)G(v)-F(v)G(u).$$

If, finally, we again write in (3) $x_1=e,\ x_2=x_3=c,\ x_4=u,\ x_5=v,\ x_6=w$, then

$$(13)\qquad f(u,v,w)=G_1(w)\big(G_1(u)G_2(v)-G_2(u)G_1(v)\big)$$

where

$$G_1(w)=G(w),\ G_2(u)=\frac{F(u)}{f(e,c,c)}.$$

Conversely, the function (13) satisfies equation (3).

Class C_{22}. In this case we have

$$f(u, c, c)=0, \ f(c, c, u)=0, \ f(c, u, c)=0 .$$

Setting $x_1=a$, $x_2=b$, $x_4=x_6=c$, $x_3=u$, $x_5=v$ in (3), we get

$$f(v, c, u)=-f(u, c, v) . \tag{14}$$

Equation (3) for $x_1=a$, $x_2=b$, $x_3=x_4=c$, $x_5=u$, $x_6=v$, becomes

$$f(c, u, v)=-f(u, v, c) . \tag{15}$$

On the basis of (14) and (15) we may conclude that if $f(a, b, c)\neq 0$, then $f(a_1, b_1, c_1)\neq 0$ where (a_1, b_1, c_1) is an arbitrary permutation of numbers a, b, c. Therefore, in (14) and (15) c may be substituted either by a or by b, so that we have seriatim

$$f(a, b, c)=-f(b, c, a)=f(c, a, b)=-f(a, b, c) .$$

Thus $f(a, b, c)=0$, which is in contradiction with $f(a, b, c)\neq 0$.

From the above it ensues that in the class C_{22} the only solution is

$$f(x, y, z)\equiv 0.$$

As $C_1 \cup C_{21} \cup C_{22}$ is the set of all functions f which are solutions of (3), we have the following theorem.

Theorem. *The general system of solutions of the functional equation* (3) *is given by* (12) *and* (13), *where* $F_1, F_2, G_1, G_2 \colon R\to R$ *are arbitrary functions.*

REFERENCE

1. D. S. Mitrinović, S. B. Prešić: *Sur une équation fonctionnelle cyclique d'ordre supérieur*. These Publications № **70** — № **76** (1962), 1—2.

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