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337. ON SOME FUNCTIONAL EQUATIONS*

Petar M. Vasić and *Radovan R. Janić*

0. Introduction

In this paper we shall determine the general solutions of the following functional equations

$$f(x_1)g(x_2, x_3, x_4)+f(x_2)g(x_3, x_4, x_1)+f(x_3)g(x_4, x_1, x_2)+f(x_4)g(x_1, x_2, x_3)=0; \tag{1}$$

$$f(x_1)g(x_2, x_3, x_4)-f(x_2)g(x_3, x_4, x_1)+f(x_3)g(x_4, x_1, x_2)-f(x_4)g(x_1, x_2, x_3)=0; \tag{2}$$

$$f(x_1)g(x_2, x_3, x_4)+f(x_3)g(x_4, x_1, x_2)=0; \tag{3}$$

$$f(x_1)g(x_2, x_3, x_4)-f(x_3)g(x_4, x_1, x_2)=0; \tag{4}$$

$$f(x_1)g(x_2, x_3, x_4)+f(x_2)g(x_3, x_4, x_1)=0; \tag{5}$$

$$f(x_1)g(x_2, x_3, x_4)-f(x_2)g(x_3, x_4, x_1)=0, \tag{6}$$

where $f: R \to R$, $g: R^3 \to R$ are the unknown functions.

Also, we shall give the general solution of the functional equation

$$a_1 f(x_1)g(x_2, x_3, x_4)+a_2 f(x_2)g(x_3, x_4, x_1)+a_3 f(x_3)g(x_4, x_1, x_2)+a_4 f(x_4)g(x_1, x_2, x_3)=0, \tag{7}$$

where a_i $(i=1, 2, 3, 4)$ are real numbers, and $f: R \to R$, $g: R^3 \to R$ unknown functions.

1. Functional equation (1)

If we exclude the trivial cases $f(x) \not\equiv 0$ and $g(x, y, z) \not\equiv 0$, then there must be four real numbers a, b, c, d such that $f(a) \neq 0$, $g(b, c, d) \neq 0$.

Putting $x_1 = x_2 = x_3 = x_4 = a$, from (1) we see that

$$g(a, a, a)=0. \tag{8}$$

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For $x_1=x_2=a$, putting $f(a)=\lambda$, from (1) we obtain

$$\lambda g(a, x_3, x_4)+\lambda g(x_3, x_4, a)+f(x_3)g(x_4, a, a)+f(x_4)g(a, a, x_3)=0. \tag{9}$$

From (9), for $x_3=a$, having in view (8), we get

$$g(a, a, x_4)+g(a, x_4, a)+g(x_4, a, a)=0. \tag{10}$$

If we put $x_1=x_3=a$ and if we introduce the notations

$$g(u, a, v)=K(u, v); \qquad g(a, u, a)=-2\lambda H(u), \tag{11}$$

(1) yields

$$\begin{aligned} K(x_2, x_4)-f(x_2)H(x_4)-f(x_4)H(x_2)& \\ +K(x_4, x_2)-f(x_4)H(x_2)-f(x_2)H(x_4)&=0, \end{aligned} \tag{12}$$

which is a cyclic equation, so that its general solution is

$$K(u, v)=g(u, a, v)=A(u, v)-A(v, u)+f(u)H(v)+f(v)H(u), \tag{13}$$

where $A: R^2\to R$ is an arbitrary function.

Setting $x_1=x_2=a$, with the notation $g(u, a, a)=-\lambda L(u)$, the equation (1), because of (10) and (11), gives

$$g(a, x_3, x_4)=-g(x_3, x_4, a)+f(x_3)L(x_4)-f(x_4)L(x_3)-2f(x_4)H(x_3). \tag{14}$$

With $x_1=a$, taking into account (13) and (14), we have from (1)

$$\begin{aligned} \lambda g(x_2, x_3, x_4)=&-f(x_2)S(x_3, x_4)+f(x_4)S(x_2, x_3) \\ &+f(x_3)\big(A(x_4, x_2)-A(x_2, x_4)\big)-f(x_2)f(x_3)H(x_4) \\ &+f(x_3)f(x_4)H(x_2)-f(x_2)f(x_4)L(x_3)+f(x_3)f(x_4)L(x_2) \end{aligned} \tag{15}$$

with $S(u, v)=g(u, v, a)$.

Finally, if we introduce the notations

$$\begin{aligned} f(u)&=F(u), \\ G_1(u, v)&=-\frac{1}{\lambda}\big(S(u, v)+F(v)L(u)\big), \\ G_2(u, v)&=-\frac{1}{\lambda}\big(A(u, v)+F(v)H(u)\big) \end{aligned}$$

we obtain

$$\begin{aligned} &f(u)=F(u), \\ &g(u, v, w)=F(u)G_1(v, w)-F(w)G_1(u, v)+F(v)\big(G_2(w, u)-G_2(u, w)\big). \end{aligned} \tag{16}$$

On the other hand, all functions of the form (16) satisfy the equation (1). Thus we have:

Theorem 1. *The general solution of the functional equation* (1) *is given by* (16), *where* $F: R\to R$, G_1, $G_2: R^2\to R$ *are arbitrary functions.*

2. Functional equation (2)

In this case we can also suppose that there are four real numbers a b, c, d so that $f(a)\neq 0$, $g(b, c, d)\neq 0$.

If we put $x_1=x_3=a$ and if we introduce the notations $g(x_2, a, x_4)=$ $=K(x_2, x_4)$, $g(a, x_4, a)=2\lambda H(x_4)$, equation (2) is reduced to the equation (12) whose general solution is determined by (13).

If we substitute $x_1=x_2=a$ in (2), we get

$$(17)\quad g(a, x_3, x_4)=g(x_3, x_4, a)-f(x_3)L(x_4)-f(x_4)L(x_3) +2H(x_3)f(x_4)+\mu f(x_3)f(x_4),$$

wherein we have inserted notations $g(u, a, a)=\lambda L(u)$, $g(a, a, a)=\lambda\mu$ and have used the equality $g(a, a, u)=g(a, u, a)-g(u, a, a)+\mu f(u)$ which is obtainable from (2) for $x_1=x_2=x_3=a$, $x_4=u$.

Now, if we again put $x_1=a$ in (2), then because of (13) and (17) we obtain

$$(18)\quad \begin{aligned}\lambda g(x_2, x_3, x_4)=&f(x_2)S(x_3, x_4)+f(x_4)S(x_2, x_3)-f(x_3)\big(A(x_4, x_2)-A(x_2, x_4)\big)\\ &-f(x_2)f(x_3)H(x_4)+f(x_3)f(x_4)H(x_2)\\ &-f(x_3)f(x_4)L(x_2)-f(x_2)f(x_4)L(x_3)+\mu f(x_2)f(x_3)f(x_4),\end{aligned}$$

with $S(u, v)=g(u, v, a)$.

Finally, introducing notations

$$f(u)=F(u),$$

$$G_1(u, v)=\frac{1}{\lambda}\big(S(u, v)-F(v)L(u)\big),$$

$$G_2(u, v)=\frac{1}{\lambda}\big(A(v, u)+F(u)H(v)\big),$$

$$\lambda\alpha=\mu,$$

we arrive at the following theorem:

Theorem 2. *The general solution of functional equation* (2) *is given by*

$$(19)\quad \begin{aligned}&f(u)=F(u),\\ &g(u, v, w)=F(u)G_1(v, w)+F(w)G_1(u, v)\\ &\qquad -F(v)\big(G_2(w, u)-G_2(u, w)\big)+\alpha F(u)F(v)F(w)\end{aligned}$$

where $F: R\to R$, G_1, $G_2: R^2\to R$ are arbitrary functions, $\alpha\in R$ is an arbitrary constant.

3. Functional equation (3)

Let us assume that $f(a)=\lambda\neq 0$. Then, for $x_1=a$, from (3) we get

$$(20)\quad g(x_2, x_3, x_4)=f(x_3)M(x_4, x_2),$$

where $M(u, v) = -\frac{1}{\lambda} g(u, a, v)$.

Substituting (20) in (3) we arrive at

$$f(x_1)f(x_3)M(x_4, x_2) + f(x_3)f(x_1)M(x_2, x_4) = 0. \tag{21}$$

For $x_1 = x_3 = a$ equation (21) yields

$$M(x_2, x_4) + M(x_4, x_2) = 0,$$

whose general solution is

$$M(x_2, x_4) = G(x_2, x_4) - G(x_4, x_2),$$

where $G: R^2 \to R$ is an arbitrary function.

Therefore, with a notation $f(x) = F(x)$, we get

Theorem 3. *The general solution of the functional equation* (3) *is given by*

$$\begin{aligned} f(u) &= F(u), \\ g(u, v, w) &= F(v)\big(G(w, u) - G(u, w)\big), \end{aligned} \tag{22}$$

where $F: R \to R$ and $G: R^2 \to R$ are arbitrary functions.

4. Functional equation (4)

If $f(a) = \lambda \neq 0$, then the equation (4) for $x_1 = a$ yields

$$g(x_2, x_3, x_4) = f(x_3)M(x_4, x_2), \tag{23}$$

where $M(u, v) = \frac{1}{\lambda} g(u, a, v)$.

Substituting (22) in (4) we get

$$f(x_1)f(x_3)M(x_4, x_2) - f(x_1)f(x_3)M(x_2, x_4) = 0,$$

wherefrom for $x_1 = x_3 = a$, we have

$$M(x_2, x_4) - M(x_4, x_2) = 0.$$

The general solution of this equation is

$$M(u, v) = G(u, v) + G(v, u), \tag{24}$$

where $G: R^2 \to R$ is an arbitrary function.

If we put $f(x) = F(x)$, in virtue of (23) and (24) we have:

Theorem 4. *The general solution of the functional equation* (4) *is*

$$\begin{aligned} f(u) &= F(u), \\ g(u, v, w) &= F(v)\big(G(u, w) + G(w, u)\big), \end{aligned} \tag{25}$$

where $F: R \to R$ and $G: R^2 \to R$ are arbitrary functions.

5. Functional equation (5)

If $f(a)=\lambda\neq 0$, then for $x_1=a$, from (5) we obtain $g(a, a, a)=0$. For $x_1=a$, the equation (5) is reduced to

$$g(x_2, x_3, x_4)=f(x_2)M(x_3, x_4), \tag{26}$$

where $M(u, v)=\frac{1}{\lambda}g(u, v, a)$.

Inserting (26) in (5) we have

$$f(x_1)f(x_2)M(x_3, x_4)+f(x_2)f(x_3)M(x_4, x_1)=0, \tag{27}$$

wherefrom, for $x_1=x_2=a$, we get

$$M(x_3, x_4)=f(x_3)N(x_4), \tag{28}$$

where $N(u)=\frac{1}{\lambda}M(u, a)=\frac{1}{\lambda^2}g(u, a, a)$.

Therefore, in virtue of (28) from (27) we get

$$f(x_1)f(x_2)f(x_3)N(x_4)+f(x_2)f(x_3)f(x_4)N(x_1)=0. \tag{29}$$

Equation (29), for $x_1=x_2=x_3=a$, $x_4=u$, yields

$$\lambda N(u)+f(u)N(a)=0.$$

Since $N(a)=\frac{1}{\lambda^2}g(a, a, a)=0$, we have

$$N(u)\equiv 0, \tag{30}$$

on basis of (30), (28) and (26) we may conclude that

$$g(u, v, w)\equiv 0.$$

Thus, the following theorem is valid:

Theorem 5. *The general solution of the functional equation* (5) *is*

$$\begin{aligned} f(u)&=F(u),\\ g(u, v, w)&\equiv 0, \end{aligned} \tag{31}$$

where $F: R\to R$ is an arbitrary function.

6. Functional equation (6)

Let $f(a)=\lambda\neq 0$. Then, for $x_1=a$, equation (6) becomes

$$g(x_2, x_3, x_4)=f(x_2)M(x_3, x_4), \tag{32}$$

in which case the notation

$$M(u, v)=\frac{1}{\lambda}g(u, v, a)$$

is introduced.

If (32) is introduced in (6) then the following equation is obtained

$$(33)\qquad f(x_1)f(x_2)M(x_3, x_4)-f(x_2)f(x_3)M(x_4, x_1)=0,$$

which, for $x_1=x_2=a,\ x_3=u,\ x_4=v$ yields

$$(34)\qquad M(u, v)=f(u)N(v),$$

where $N(u)=\frac{1}{\lambda}M(u, a)=\frac{1}{\lambda^2}g(u, a, a)$.

On basis of (34), (33) becomes

$$f(x_1)f(x_2)f(x_3)N(x_4)-f(x_2)f(x_3)f(x_4)N(x_1)=0,$$

which, for $x_1=x_2=x_3=a,\ x_4=u$, gives

$$(35)\qquad N(u)=\mu f(u)$$

where $\mu=\frac{1}{\lambda}N(a)=\frac{1}{\lambda^3}g(a, a, a)$.

Using (32), (34) and (35) we find

$$g(u, v, w)=\mu f(u)f(v)f(w).$$

Accordingly, the following theorem is valid:

Theorem 6. *The general solution of the equation* (6) *is given by*

$$(36)\qquad \begin{aligned} f(u)&=F(u),\\ g(u, v, w)&=\mu F(u)F(v)F(w), \end{aligned}$$

where $F: R\to R$ is an arbitrary function and $\mu\in R$ an arbitrary constant.

7. Functional equation (7)

If we put $f(x)g(y, z, t)=h(x, y, z, t)$ the equation (7) gives

$$a_1h(x_1, x_2, x_3, x_4)+a_2h(x_2, x_3, x_4, x_1)$$
$$+a_3h(x_3, x_4, x_1, x_2)+a_4h(x_4, x_1, x_2, x_3)=0.$$

Using the result of B. Zarić (see [1]) we have the following

Lemma 1. *Functional equation* (7) *is equivalent to*:

1° $f(x_1)g(x_2, x_3, x_4)=0$

if $(a_1+a_3)^2-(a_2+a_4)^2\neq 0$ *and* $(a_1-a_3)^2+(a_2-a_4)^2\neq 0$;

2° $f(x_1)g(x_2, x_3, x_4)+f(x_2)g(x_3, x_4, x_1)$

$$+f(x_3)g(x_4, x_1, x_2)+f(x_4)g(x_1, x_2, x_3)=0$$

if $a_1=a_2=a_3=a_4\neq 0$;

3° $f(x_1)g(x_2, x_3, x_4)-f(x_2)g(x_3, x_4, x_1)$

$$+f(x_3)g(x_4, x_1, x_2)-f(x_4)g(x_1, x_2, x_3)=0$$

if $(a_1-a_3)^2+(a_2-a_4)^2\neq 0$ *and* $a_2+a_4=-(a_1+a_3)\neq 0$;

4° $f(x_1)g(x_2, x_3, x_4)+f(x_3)g(x_4, x_1, x_2)=0$

if $(a_1+a_3)^2-(a_2+a_4)^2\neq 0$, $a_1=a_3$, $a_2=a_4$;

5° $f(x_1)g(x_2, x_3, x_4)-f(x_3)g(x_4, x_1, x_2)=0$

if $a_2+a_4=0$, $a_1+a_3=0$, $a_1^2+a_2^2\neq 0$;

6° $f(x_1)g(x_2, x_3, x_4)+f(x_2)g(x_3, x_4, x_1)=0$

if $a_1+a_3=a_2+a_4\neq 0$, $(a_1-a_3)^2+(a_2-a_4)^2\neq 0$;

7° $f(x_1)g(x_2, x_3, x_4)-f(x_2)g(x_3, x_4, x_1)=0$

if $(a_1-a_3)^2+(a_2-a_4)^2\neq 0$, $a_2+a_4=-(a_1+a_3)\neq 0$;

8° $0=0$

if $a_1=a_2=a_3=a_4=0$.

Combining Lemma 1 and Theorems 1—6 we get the following:

Theorem 7. *The general solution of the functional equation* (7) *is determined*

1° *by* $f(u)=0$, $g(u, v, w)$ *arbitrary or* $f(u)$ *arbitrary* $g(u, v, w)=0$ *if*

$$(a_1+a_3)^2-(a_2+a_4)^2\neq 0 \text{ and } (a_1-a_3)^2+(a_2-a_4)^2\neq 0,$$

2° *by the formula* (16) *if* $a_1=a_2=a_3=a_4\neq 0$,

3° *by the formula* (19) *if* $(a_1-a_3)^2+(a_2-a_4)^2\neq 0$, $a_2+a_4=-(a_1+a_3)\neq 0$,

4° *by the formula* (22) *if* $(a_1+a_3)^2+(a_2+a_4)^2\neq 0$, $a_1=a_3$, $a_2=a_4$,

5° *by the formula* (25) *if* $a_1+a_3=0$, $a_2+a_4=0$, $a_1^2+a_2^2\neq 0$,

6° *by the formula* (31) *if* $a_1+a_3=a_2+a_4\neq 0$, $(a_1-a_3)^2+(a_2-a_4)^2\neq 0$,

7° *by the formula* (36) *if* $(a_1-a_3)^2+(a_2-a_4)^2\neq 0$, $a_1+a_3=-(a_2+a_4)\neq 0$,

8° *by* $f(x)=F(x)$, $g(x, y, z)=G(x, y, z)$, *where* $F: R\to R$, $G: R^3\to R$ *are arbitrary functions if* $a_1=a_2=a_3=a_4=0$.

Remark. Functional equations (1) — (7) represent generalizations in the form of a number of up to now considered functional equations. So, for example, functional equation (2) represents the generalization of functional equation

$$(37)\qquad f(x_1, x_2, x_3)f(x_4, x_5, x_6)-f(x_1, x_2, x_4)f(x_5, x_6, x_3)$$
$$+f(x_1, x_2, x_5)f(x_6, x_3, x_4)-f(x_1, x_2, x_6)f(x_3, x_4, x_5)=0$$

whose general solution was determined by P. M. VASIĆ in paper [2]. However, it is necessary to notice that the general solution of this equation cannot be directly obtained from the general solution of equation (2). It seems that the transfer from general solution of equation (2) to general solution of equation (37) is even more complicated than the solving of equation (37).

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Katedra za matematiku
Elektrotehnički fakultet
Beograd, Jugoslavija