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# 333. ON SOME GENERALIZATIONS OF THE PEXIDER EQUATIONS*

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Let $A_1, \ldots, A_k$, $k \geqq 2$, be any sets; let $G_1$ with the operation $*$ be an semigroup with the neutral element $e$, and let $G_2$ with the operation $\circ$ be an Abelian group. Let us assume that there are $k$ functions $\Phi_1, \ldots, \Phi_k$:

$$\Phi_i : A_i \xrightarrow{\text{onto}} G_1, \quad i = 1, \ldots, k, \tag{1}$$

and let us consider the functional equation

$$f[\Phi_1(x_1) * \cdots * \Phi_k(x_k)] = g_1(x_1) \circ \cdots \circ g_k(x_k), \qquad x_i \in A_i, \; i = 1, \ldots, k, \tag{2}$$

where the functions

$$f : G_1 \to G_2,$$

$$g_i : A_i \to G_2, \quad i = 1, \ldots, k,$$

are unknown.

**Theorem 1.** *If the functions $f, g_1, \ldots, g_k$ satisfy equation* (2), *then there exist constants $a_1, \ldots, a_k \in G_2$ and a function $\varphi : G_1 \to G_2$ satisfying the equation*

$$\varphi(x * y) = \varphi(x) \circ \varphi(y) \tag{3}$$

*such that*

$$f(x) = a_1 \circ \cdots \circ a_k \circ \varphi(x), \quad x \in G_1, \tag{4}$$

$$g_i(x_i) = a_i \circ \varphi[\Phi_i(x_i)], \quad x_i \in A_i, \; i = 1, \ldots, k. \tag{5}$$

*Conversely, if the functions $f, g_1, \ldots, g_k$ are defined by* (4) *and* (5), *where $a_1, \ldots, a_k$ belong to $G_2$ and the function $\varphi : G_1 \to G_2$ satisfies equation* (3), *then $f, g_1, \ldots, g_k$ satisfy equation* (2).

***Proof.*** It follows from (1) that for any $1 \leqq i \leqq k$ there exists an $x_i^0 \in A_i$ such that

$$\Phi_i(x_i^0) = e. \tag{6}$$

Let

$$a_i = g_i(x_i^0), \quad i = 1, \ldots, k. \tag{7}$$

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Putting in (2)

$$x_1=\overset{0}{x}_1,\ \ldots,\ x_{i-1}=\overset{0}{x}_{i-1},\ x_{i+1}=\overset{0}{x}_{i+1},\ \ldots,\ x_k=\overset{0}{x}_k,\quad i=1,\ldots,k,$$

and taking to consideration (6) and (7) we can see that

$$g_i(x_i)=a_1^{-1}\circ\cdots\circ a_{i-1}^{-1}\circ a_{i+1}^{-1}\circ\cdots\circ a_k^{-1}\circ f[\Phi_i(x_i)],\quad i=1,\ldots,k. \tag{8}$$

Inserting expressions (8) into equation (2) and writing

$$b=a_1^{-1}\circ\cdots\circ a_k^{-1} \tag{9}$$

we obtain the following equation

$$f[\Phi_1(x_1)*\cdots*\Phi_k(x_k)]=\underbrace{b\circ\cdots\circ b}_{(k-1)\text{ times}}\circ f[\Phi_1(x_1)]\circ\cdots\circ f[\Phi_k(x_k)].$$

After introducing the following symbols

$$y_i=\Phi_i(x_i),\quad x_i\in A_i,\quad i=1,\ldots,k, \tag{10}$$

$$\varphi(x)=b\circ f(x),\quad x\in G_1, \tag{11}$$

the last equation takes the form

$$\varphi(y_1*\cdots*y_k)=\varphi(y_1)\circ\cdots\circ\varphi(y_k). \tag{12}$$

Putting $x_i=\overset{0}{x}_i$, $i=1,\ldots,k$ in equation (2) and using relations (6) and (7) we obtain

$$f(e)=a_1\circ\cdots\circ a_k,$$

whence it follows in view of (9) and (11) that $\varphi(e)$ is the neutral element of the group $G_2$. Putting

$$y_1=x,\ y_2=y,\ y_j=e\ \text{for}\ j=3,\ldots,k$$

into equation (12) we obtain equation (3). Relation (4) results from (11) and (9), and from (8) and (4) we have (5).

Applying the principle of the mathematical induction we confirm that the function $\varphi: G_1\to G_2$ which satisfies equation (3), satisfies equation (12) as well, and hence the functions $f, g_1,\ldots,g_k$ which are defined by expressions (4) and (5), where $a_1,\ldots,a_k\in G_2$ and the function $\varphi: G_1\to G_2$ is a solution of equation (3), satisfy equation (2).

In some particular cases we may omit the assumption that the set $G_2$ with the operation $\circ$ is a group. Let us namely consider the equation

$$f[\Phi_1(x_1)*\cdots*\Phi_k(x_k)]=g_1(x_1)\cdot\cdots\cdot g_k(x_k), \tag{13}$$

$$x_i\in A_i,\ i=1,\ldots,k,$$

where $\Phi_i$, $i=1, \dots, k$ are given functions satisfying (1), and

$$f: G_1 \to R,$$

$$g_i: A_i \to R, \quad i=1, \dots, k,$$

are unknown functions.

**Theorem 2.** *The solutions of equation* (13) *have the form*

$$\text{(I)} \qquad \begin{cases} f(x) = \prod_{i=1}^{k} a_i \cdot \varphi(x), & x \in G_1, \\ g_i(x_i) = a_i \cdot \varphi[\Phi_i(x_i)], & x_i \in A_i, \ i=1, \dots, k, \end{cases}$$

*or*

$$\text{(II)} \qquad \begin{cases} f(x) \equiv 0, \\ g_j(x_j) \equiv 0, \\ g_1, \dots, g_{j-1}, g_{j+1}, \dots, g_k \text{ — arbitrary,} \end{cases}$$

*where* $\varphi: G_1 \to R$ *is a solution of the equation*

$$\varphi(x * y) = \varphi(x) \cdot \varphi(y),$$

*and* $a_i$, $i=1, \dots, k$, *are real constants different from zero.*

***Proof.*** Making the same substitutions as in the proof of theorem 1 we obtain

$$f[\Phi_i(x_i)] = \prod_{\substack{j=1 \\ j \neq i}}^{k} a_j \cdot g_i(x_i), \quad i=1, \dots, k. \tag{14}$$

Now let us consider two cases.

1° Each $a_i$ for $i=1, \dots, k$ is different from zero. We repeat now the argument from the proof of theorem 1 and we obtain system (I) as the solution of equation (13).

2° At least one of the $a_i$ $(i=1, \dots, k)$ is equal to zero. Hence and from (14) we draw the conclusion that the function $f$ is identically equal to zero, i.e. the solution of equation (13) is given by system (II).

In the particular case, where $k=2$, $A_1 \subset R^n$, $A_2 \subset R^m$ and $G_1 = G_2$ is the additive group of real numbers, equation (2) was considered by I. KOTLARSKI in paper [2]. The method of the proof of theorem 1 is similar to the method which is employed in the paper [2].

**Remark.** Taking $k=2$ and $A_1=A_2=G_1=R$, and assuming that the functions $\Phi_1$ and $\Phi_2$ are identities in $R$, and $G_2$ is the additive g-oup of real numbers and that operation $*$ is either the addition or the multiplication of numbers, we obtain from (2) and (13) the following equations

$$f(x+y)=g_1(x)+g_2(y),$$

$$f(x\cdot y)=g_1(x)+g_2(y), \tag{2'}$$

$$f(x+y)=g_1(x)\cdot g_2(y),$$

$$f(x\cdot y)=g_1(x)\cdot g_2(y), \tag{13'}$$

which are called the PEXIDER equations (see [1], p. 116).

## REFERENCES

1. J. ACZÉL: *Vorlesungen über Funktionalgleichungen und ihre Anwendungen.* Stuttgart 1961.

2. I. KOTLARSKI: *On some generalization of the Pexider equation.* Aequationes Math. (to appear)

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