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332. THE INEQUALITIES OF RADO AND POPOVICIU*

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This note is a brief survey of certain inequalities of Rado and Popoviciu type, with an attempt at a unified treatment using convex functions.

This note is a brief survey of several recent generalizations of the inequalities of RADO and POPOVICIU (inequalities (17) and (19) below). Besides being a survey this note attempts to give these results a unified treatment by the use of convex functions. No claim is made for originality as all of these results have appeared elsewhere in the references given in the bibliography.

Inequalities that can be shown to be special cases of some relatively simple general inequality do not thereby lose their own importance or interest. A very obvious classical case is that the much more important and useful HÖLDER inequality is a special case of the simpler and more elegant inequality of YOUNG.

Unification by the use of convex functions is not of course the only one possible. Several other methods have been used to get these results. Elementary calculus is one — as in section 3 below. Another is to show that that the given inequality can be written in a form that exhibits it as a special case of a simpler known inequality. In **[2]** both these methods occur, giving alternative proofs of the same result. A completely different method has been devised by MITRINOVIĆ and VASIĆ. Unlike the other methods mentioned, this method is creative in that it allows for the discovery as well as the proof of new inequalities; it is given an elegant exposition in **[18]**.

0. Functions in this note will always be real valued functions of a real variable. If f is a function, $(a)=\{a_1, a_2, \ldots\}$ a sequence of real numbers then $f(a)$ will denote the sequence $\{f(a_1), f(a_2), \ldots\}$. If (w) is a sequence of positive numbers write $W_n=\sum_{k=1}^{n} w_k$.

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The following notations are standard.

$$\begin{aligned} M_n^{[r]}(a;w) &= \left(\frac{1}{W_n}\sum_{k=1}^{n} a_k^{\,r} w_k\right)^{1/r}, && r\neq 0,\quad |r|<+\infty,\\ &= \left(\prod_{k=1}^{n} a_k^{w_k}\right)^{1/W_n} = G_n(a;w), && r=0,\\ &= \max(a_1,\dots,a_n), && r=+\infty,\\ &= \min(a_1,\dots,a_n), && r=-\infty,\\ A_n(a;w) &= M_n^{[1]}(a;w). \end{aligned}$$

More generally if F is any strictly monotonic function,

$$\mathfrak{F}_n(a;w) = F\left(\frac{1}{W_n}\sum_{k=1}^{n} F^{-1}(a_k)\,w_k\right). \tag{1}$$

By choosing $F(x)=x^{1/r}$, or e^x these general means reduce to those defined above.

If I is any set of positive integers then define $W_I=\sum_{k\in I} w_k$ and

$$\mathfrak{F}_I(a;w) = F\left(\frac{1}{W_I}\sum_{k\in I} F^{-1}(a_k)\,w_k\right).$$

If $w_1=\cdots=w_n$ these means will be written $M_n^{[r]}(a)$, etc.; if there is no ambiguity a and, or w will be omitted. It should be remarked that the choice of F in (1) often places a restriction on (a); thus if $F(x)=x^{1/r}$ or e^x, (a) must be taken as a sequence of positive numbers. When a particular F is chosen this restriction will be stated, otherwise it is always implied that the sequence must be such that F, and F^{-1}, are defined wherever they occur.

We need certain properties of continuous convex functions; in particular if f is such a function and $\sum_{k=1}^{n} b_k=1$, $b_k\geq 0$, $k=1,\dots,n$, then

$$f\left(\sum_{k=1}^{n} a_k b_k\right)\leq \sum_{k=1}^{n} b_k f(a_k), \tag{2}$$

and if f is strictly convex inequality (2) is strict unless $a_1=\cdots=a_n$. The main properties of such functions can be found in **[1, 15, 18]**; if a continuous function f is both convex and concave it is linear.

1. The relationship between various means of type (1) can easily be given using the properties of convex functions, [**15**, Theorem 92].

Theorem 1. *If F, G are two strictly monotonic continuous functions, G increasing and $G^{-1}\circ F$ convex then*

$$\mathfrak{F}_n(a;w)\leq \mathfrak{G}_n(a;w). \tag{3}$$

If $G^{-1}\circ F$ is strictly convex inequality (3) *is strict unless $a_1=\cdots=a_n$. If either G is decreasing or $G^{-1}\circ F$ is concave inequality* (3) *is reversed.*

Proof. This is an immediate consequence of (2).

Remarks. (i) Particular cases of (3) give

(4) $$M_n^{[r]}(a;\, w) \leqq M_n^{[s]}(a;\, w), \qquad -\infty \leqq r < s \leqq +\infty,$$

for any positive sequence (a), with equality if and only if $a_1 = \cdots = a_n$.

(ii) If $r=0$, $s=1$ $w_1 = \cdots = w_n$ then inequality (4) is the famous arithmetic-geometric mean inequality,

(5) $$G_n \leqq A_n.$$

(iii) By simple algebraic manipulations (4) can be generalized to allow the two means to have different weights, [**4, 5, 19**].

(iv) Inequality (3) is equivalent to

(6) $$G^{-1}(\mathfrak{F}_n(a;\, w)) \leqq A_n(G^{-1}(a);\, w).$$

But, as the proof of Theorem 1 shows, (6) does not require G to be increasing.

(v) Using remark (iv) choose $F(x) = x^{1/r}$, $G^{-1}(x) = \dfrac{1}{1+x^s}$ then if $x^{s/r} \geqq \dfrac{s-r}{s+r}$, $s \geqq r > 0$, $G^{-1} \circ F$ is strictly convex and (6) implies that

(7) $$\sum_{k=1}^{n} \frac{w_k}{1+a_k^{\,s}} \geqq \frac{W_n}{1+(M_n^{[r]}(a;\, w))^s},$$

provided $a_k^{\,s} \geqq \dfrac{s-r}{s+r}$, $1 \leqq k \leqq n$. If instead $F(x) = e^x$ then (7) can be extended to the case $r=0$. This inequality is a generalization of one due to Henrici, [**8, 23**].

2. Inequality (5) can be rewritten $A_n - G_n \geqq 0$, and in this form Rado obtained a better lower bound proving, [**14**, Theorem 60]

(8) $$A_n - G_n \geqq \frac{1}{n} \max_{1 \leqq j,\, k \leqq n} \left(\sqrt{a_k} - \sqrt{a_j}\right)^2.$$

Popoviciu, [**25**], proved that if (5) is written $A_n/G_n \geqq 1$ then the lower bound can be improved to give

(9) $$\frac{A_n}{G_n} \geqq \max_{1 \leqq j,\, k \leqq n} \left\{\frac{1}{2}\left(\sqrt{\frac{a_j}{a_k}} + \sqrt{\frac{a_k}{a_j}}\right)\right\}^{2/n}.$$

Both Rado's inequality and Popoviciu's inequality and the many generalizations due to Bullen, Mitrinović, Vasić and others, [**2, 3, 4, 5, 7, 8, 19, 20, 22, 23, 24, 25**] can be deduced from a very simple but general observation due to Everitt, [**12**].

Theorem 2. *Let* (a) *be a given sequence,* H, F *two continuous functions such that* $H \circ F$ *is convex and define*

$$\alpha(H, F, w;\, I) = \alpha(I) = W_I H(\mathfrak{F}_I(a;\, w)).$$

Then if $I \cap J = \emptyset$

(10) $$\alpha(I \cup J) \leqq \alpha(I) + \alpha(J).$$

Further if $H \circ F$ *is strictly convex inequality* (10) *is strict unless*

$$H(\mathfrak{F}_I(a;\, w)) = H(\mathfrak{F}_J(a;\, w)).$$

Proof. Using (2) this is an immediate consequence of the convexity of $H \circ F$.

Corollary 3. *Let (a) be a given sequence, (p), (q) given positive sequences, F, G, H, K four continuous functions such that $H \circ F$ is convex, and $K \circ G$ is concave and define*

$$\beta(I) = \alpha(H, F, p; I) - \alpha(K, G, q; I).$$

Then if $I \cap J = \emptyset$

$$\beta(I \cup J) \leqq \beta(I) + \beta(J). \tag{11}$$

If $H \circ F$ is strictly convex and $K \circ G$ strictly concave then inequality (11) is strict unless (i) $H(\mathfrak{F}_I(a; p)) = H(\mathfrak{F}_J(a; p))$ and (ii) $K(\mathfrak{G}_I(a; q)) = K(\mathfrak{G}_J(a; q))$. If $H \circ F$ is strictly convex and $K \circ G$ linear equality occurs in (11) only when (i) holds; and if $H \circ F$ is linear and $K \circ G$ strictly concave equality occurs in (11) only when (ii) holds.

Proof. This is an immediate consequence of Theorem 2.

Remarks. (i) Theorem 2 and Corollary 3 say that α, β are subadditive set functions on the positive integers.

(ii) The most important cases of inequality (11) occur when

$$I = \{1, 2, \dots, n-1\}; \quad J = \{n\}; \quad I \cup J = \{1, 2, \dots, n\}.$$

Corollary 4. *Under the assumptions in Corollary 3*

$$\begin{aligned} P_n H(\mathfrak{F}_n(a; p)) - Q_n K(\mathfrak{G}_n(a; q)) \\ \leqq P_{n-1} H(\mathfrak{F}_{n-1}(a; p)) - Q_{n-1} K(\mathfrak{G}_{n-1}(a; q)) + p_n H(a_n) - q_n K(a_n). \end{aligned} \tag{12}$$

Remarks. (i) The cases of equality in Corollary 4 are easily stated.

(ii) If $(p) = (q) = (w)$, $H = K$ then the last two terms on the right hand side of (12) cancel to give

$$W_n \{H(\mathfrak{F}_n(a; w) - H(\mathfrak{G}_n(a; w))\} \leqq W_{n-1} \{H(\mathfrak{F}_{n-1}(a; w) - H(\mathfrak{G}_{n-1}(a; w))\}. \tag{13}$$

(iii) Because of the symmetry and simplicity of (13) compared to (12), it is of some interest to modify Corollary 4.

Theorem 5. *Under the assumptions of Corollary 3 if μ, ν are any two real numbers*

$$\begin{aligned} Q_n \left\{H \circ F\left(\lambda A_n(F^{-1}(a); p) + \mu\right) - K \circ G\left(A_n(G^{-1}(a); q) + \nu\right)\right\} \\ \leqq Q_{n-1} \left\{H \circ F\left(\lambda' A_{n-1}(F^{-1}(a); p) + \mu'\right) - K \circ G\left(A_{n-1}(G^{-1}(a); q) + \nu'\right)\right\}, \end{aligned} \tag{14}$$

where $\lambda = \left(\dfrac{P_n q_n}{p_n Q_n}\right) \dfrac{F^{-1} \circ H^{-1} \circ K(a_n)}{F^{-1}(a_n)}$, $\lambda' = \dfrac{Q_n P_{n-1}}{P_n Q_{n-1}} \lambda$, $\mu' = \dfrac{Q_n}{Q_{n-1}} \mu$, $\nu' = \dfrac{Q_n}{Q_{n-1}} \nu$.

If $H\circ F$ is strictly convex and $K\circ G$ strictly concave inequality (14) *is strict unless*

$$(i)\ a_n = K^{-1}\circ H\circ F\left(\lambda' A_{n-1}\left(F^{-1}(a); p\right)+\mu'\right),$$

$$(ii)\ a_n = G\left(A_{n-1}\left(G^{-1}(a); q\right)+\nu'\right).$$ *If $H\circ F$ is strictly convex and $K\circ G$ linear*

then inequality (14) *is strict unless* (*i*) *holds*; *if $H\circ F$ is linear and $K\circ G$ strictly concave then inequality* (14) *is strict unless* (*ii*) *holds.*

Proof. Since $K\circ G$ is concave,

$$Q_n K\circ G\left(A_n(G^{-1}(a); q)+\nu\right) = Q_n K\circ G\left(\left(A_{n-1}(G^{-1}(a); q)+\nu'\right)\frac{Q_{n-1}}{Q_n}+G^{-1}(a_n)\frac{q_n}{Q_n}\right)$$

$$\geqq Q_{n-1}K\circ G\left(A_{n-1}(G^{-1}(a); q)+\nu'\right)+q_n K(a_n).$$

Since $H\circ F$ is convex,

$$Q_n H\circ F\left(\lambda A_n(F^{-1}(a); p)+\mu\right)$$

$$= Q_n H\circ F\left(\left(\lambda' A_{n-1}\left(F^{-1}((a); p)\right)+\mu'\right)\frac{Q_{n-1}}{Q_n}+\frac{q_n}{Q_n}F^{-1}\circ H^{-1}\circ K(a_n)\right)$$

$$\leqq Q_{n-1}H\circ F\left(\lambda' A_{n-1}(F^{-1}(a); p)+\mu'\right)+q_n K(a_n).$$

Inequality (14) is now immediate, as are the cases of equality.

Let us consider some particular cases of these results.

(a) If $x>0$ and $H(x)=x^t$, $F(x)=x^{\frac{1}{r}}$, $rt\neq 0$, then $H\circ F$ is convex provided $\frac{r}{t}\leqq 1$, (strictly convex unless $\frac{r}{t}=1$). Then Theorem 2 implies that for positive sequences (a)

$$(15)\qquad \mu(I)=W_I\left(M_I^{[r]}(a; w)\right)^t$$

is subadditive, a result due to EVERITT, and MCLAUGHLIN and METCALF, [**12, 17**]. If we take $F(x)=e^x$ then this result can be extended to the case $r=0$.

(b) If $x>0$ and $F(x)=x^{\frac{1}{r}}$, $G(x)=x^{\frac{1}{s}}$, $H(x)=x^t$, $K(x)=x^u$, $rt\neq 0$, $su\neq 0$, then if $\frac{r}{t}\leqq 1$ $H\circ F$ is convex, and if $\frac{s}{u}\geqq 1$ $K\circ G$ is concave. Then Corollary 3 implies that for positive sequences (a)

$$\mu(I)=P_I\left(M_I^{[r]}(a; p)\right)^t-Q_I\left(M_I^{[s]}(a; q)\right)^u$$

is subadditive; by taking $F(x)$ or $G(x)$ or e^x this can be extended to allow r$=0$, or $s=0$, [**3, 4, 5**].

If $x>0$, $H(x)=K(x)=\log x$, $F(x)=x^{\frac{1}{r}}$, $G(x)=x^{\frac{1}{s}}$ then $H\circ F$ is strictly convex if $r>0$, $K\circ G$ strictly concave if $s<0$. Then Corollary 3 implies that for (a) a positive sequence

$$\mu(I)=\frac{(M_I^{[r]}(a;p))^{P_I}}{(M_I^{[s]}(a;q))^{Q_I}}$$

is logarithmically subadditive; again, taking $F(x)$ or $G(x)$ to be e^x will extend this result to $r=0$, $s=0$, [3].

(c) If $x>0$, $F(x)=x^{\frac{1}{r}}$, $G(x)=x^{\frac{1}{s}}$, $H(x)=x^t$, $rst\neq 0$; then provided $\frac{r}{t}\leqq 1$, $H\circ F$ is convex (strictly if $r\neq t$) and if $\frac{s}{t}\geqq 1$, $H\circ G$ is concave (strictly if $s\neq t$). Hence if (a) is a positive sequence inequality (13) gives

$$(16)\qquad W_n\{(M_n^{[s]}(a;w))^t-(M_n^{[r]}(a;w))^t\}\geqq W_{n-1}\{(M_{n-1}^{[s]}(a;w))^t-(M_{n-1}^{[r]}(a;w))^t\}.$$

The cases of equality can easily be deduced from Theorem 2. If we take $F(x)=e^x$ then inequality (16) can be extended to the case $r=0$. [**3, 5, 7, 20**]. If $s=t=1$, $r=0$, $w_1=\cdots=w_n$ then inequality (16) reduces to Rado's inequality

$$(17)\qquad n(A_n-G_n)\geqq(n-1)(A_{n-1}-G_{n-1}),$$

with equality if and only if $a_n=G_{n-1}$. Repeated application of inequality (17) leads to (8); an interesting variation of (8), with an upper bound is given by Kober, and Diananda, [**10, 16**].

(d) If $x>0$, $F(x)=x^{\frac{1}{r}}$, $G(x)=x^{\frac{1}{s}}$, $H(x)=\log x$, $r<0<s$ then $H\circ F$ is strictly concave and $H\circ G$ strictly convex. So if (a) is a positive sequence inequality (13) implies

$$(18)\qquad \left(\frac{M_n^{[s]}(a;w)}{M_n^{[r]}(a;w)}\right)^{W_n}\geqq\left(\frac{M_{n-1}^{[s]}(a;w)}{M_{n-1}^{[r]}(a;w)}\right)^{W_{n-1}},$$

with equality if and only if $a_n=M_{n-1}^{[s]}(a;w)=M_{n-1}^{[r]}(a;w)$. If instead we take $F(x)$ or $G(x)$ or e^x, inequality (18) can be extended to the cases $r=0$, $s=0$.

If $s=1$, $r=0$, $w_1=\cdots=w_n$ then inequality (18) reduces to Popoviciu's inequality

$$(19)\qquad \left(\frac{A_n}{G_n}\right)^n\geqq\left(\frac{A_{n-1}}{G_{n-1}}\right)^{n-1},$$

with equality if and only if $a_n=A_{n-1}$. Repeated application of (19) leads to inequality (9); this last inequality also has a Kober-Diananda variant, [3].

(e) If $F(x)=x$, $H(x)=G^{-1}(x)=\frac{1}{1+\alpha(x)}$, α chosen so that H is strictly convex then (13) leads to a Rado extension of Henrici's inequality

$$\sum_{k=1}^{n}\frac{w_k}{1+\alpha(a_k)}-\frac{W_n}{1+\alpha(A_n(a;w))}\geqq\sum_{k=1}^{n-1}\frac{w_k}{1+\alpha(a_k)}-\frac{W_{n-1}}{1+\alpha(A_{n-1}(a;w))},$$

with equality only if $a_n=A_{n-1}(a;w)$; various simple choices of α are possible as we have seen in inequality (7), [**8, 23**].

(f) Starting from inequality (14) it is possible to obtain extensions of these inequalities, allowing different weights in the various means. Thus the following are two possible extensions of RADO's inequality (17), [8];

$$P_n\{A_n(a;\ p)-(G_n(a;\ q))^{\frac{p_nQ_n}{q_nP_n}}\}\geqq P_{n-1}\{A_{n-1}(a;\ p)-(G_{n-1}(a;\ q))^{\frac{p_nQ_{n-1}}{q_nP_{n-1}}}\}$$

with equality only when $a_n=G_n(a;\ q)^{\frac{p_nQ_{n-1}}{q_nP_{n-1}}}$;

$$Q_n\left\{\frac{q_nP_n}{p_nQ_n}A_n(a;\ p)-G_n(a;\ q)\right\}\geqq Q_{n-1}\left\{\frac{q_nP_{n-1}}{p_nQ_{n-1}}A_{n-1}(a;\ p)-G_{n-1}(a;\ q)\right\}$$

with equality only when $a_n=G_{n-1}(a;q)$. These follow from (14) by taking $\mu=\nu=0$, $H(x)=K(x)=x$ and $F(x)=e^x$, $G(x)=x$ for the first, $F(x)=-x$, $G(x)=-e^x$ for the second.

(g) Finally, by considering non-zero values of the parameters μ, and ν, in (14), inequalities of the MITRINOVIĆ-VASIĆ type can be obtained, [20]. Thus (14) leads to the following, that generalises (e) above.

$$(20)\quad Q_n\left\{\frac{q_nP_n}{p_nQ_n}A_n(a;\ p)-\alpha G_n(a;\ q)\right\}\geqq Q_{n-1}\left\{\frac{q_nP_{n-1}}{p_nQ_{n-1}}A_{n-1}(a;\ p)-\alpha^{\frac{p_n}{P_{n-1}}}G_{n-1}(a;\ q)\right\}$$

for $\alpha>0$, and with equality only when $a_n=\alpha^{\frac{p_n}{P_{n-1}}}G_{n-1}(a;\ q)$. This follows from (14) by taking $H(x)=K(x)=x$, $F(x)=-x$, $G(x)=-e^x$, $\mu=0$, $\alpha=e^\nu$.

Before leaving this section it should be noted that certain inequalities between sums lead to inequalities between means that are not deducible in this way from properties of convex functions: several such inequalities are due to MITRINOVIĆ and VASIĆ, [5, 17, 19]. The following result is perhaps the simplest and most general; it is due to MITRINOVIĆ and VASIĆ, [24].

Theorem 6. *If* λ, $\mu>0$, $\lambda+\mu\geqq 1$, (a), (b), (p), (q) *positive sequences,* I_1, I_2, J_1, J_2 *non-empty sets of integers with* $I_1\cap J_1=I_2\cap J_2=\emptyset$, *then*

$$(21)\qquad P^\lambda_{I_1\cup J_1}Q^\mu_{I_2\cup J_2}(M^{[r]}_{I_1\cup J_1}(a;\ p))^{\lambda r}(M^{[s]}_{I_2\cup J_2}(b;\ q))^{\mu s}$$

$$\geqq P^\lambda_{I_1}Q^\mu_{I_2}(M^{[r]}_{I_1}(a;\ p))^{\lambda r}(M^{[s]}_{I_2}(b;\ q))^{\mu s}+P^\lambda_{J_1}Q^\mu_{J_2}(M^{[r]}_{J_1}(a;\ p))^{\lambda r}(M^{[s]}_{J_2}(b;\ q))^{\mu s}.$$

If $\lambda+\mu>1$, (21) *is strict; if* $\lambda+\mu=1$, (21) *is strict unless*

$$P_{I_1}Q_{J_2}(M^{[r]}_{I_1}(a;\ p))^r(M^{[s]}_{J_2}(b;\ q))^s=P_{J_1}Q_{I_2}(M^{[r]}_{J_1}(a;\ p))^r(M^{[s]}_{I_2}(b;\ q))^s.$$

Proof. This is an immediate consequence of the inequality

$$a_1^\lambda b_1^\mu+a_2^\lambda b_2^\mu\leqq(a_1+a_2)^\lambda(b_1+b_2)^\mu$$

and the conditions under which it is strict, [13, p. 29].

Remarks. (i) Many particular cases of this inequality are given in **[17, 24]**. In particular the subadditivity of the function μ defined in (15) can be obtained from (21); see **[24]**, where a more general result is given.

(ii) If $rs<0$, $(a)=(b)$, choosing $\lambda=\frac{s}{s-r}$, $\mu=-\frac{r}{s-r}$ $I=\{1,\dots,n-1\}$, $J=\{n\}$, then (21) gives the following result of Mitrinović and Vasić, **[5, 19, 22]**.

$$\frac{P_n^{\frac{s}{s-r}}}{Q_n^{\frac{r}{s-r}}}\left(\frac{M_n^{[r]}(a;p)}{M_n^{[s]}(a;q)}\right)^{\frac{rs}{s-r}} \geqq \frac{P_{n-1}^{\frac{s}{s-r}}}{Q_{n-1}^{\frac{r}{s-r}}}\left(\frac{M_{n-1}^{[r]}(a;p)}{M_{n-1}^{[s]}(a;q)}\right)^{\frac{rs}{s-r}} + \frac{p_n^{\frac{s}{s-r}}}{q_n^{\frac{r}{s-r}}}.$$

3. A simple direct proof of Rado's inequality, (17); can be given as follows. Let $a_n=x$ and put

$$f(x)=A_n-G_n=\frac{n-1}{n}A_{n-1}+\frac{x}{n}-G_{n-1}^{\frac{n-1}{n}}x^{\frac{1}{n}}.$$

Then

$$f'(x)=\frac{1}{n}\left(1-\left(\frac{G_{n-1}}{x}\right)^{\frac{n-1}{n}}\right);$$

that is to say, f has a single minimum at $x=G_{n-1}$. In other words unless $a_n=G_{n-1}$.

$$f(a_n)=A_n-G_n>f(G_{n-1})=\frac{n-1}{n}(A_{n-1}-G_{n-1}).$$

Remark. This method of proof has been used to obtain many of the inequalities in the previous section, **[2, 19, 20, 21, 23]**.

Rewriting inequality (17) as $\frac{A_n-G_n}{A_{n-1}-G_{n-1}}>\frac{n-1}{n}$, the above argument shows that the left-hand side has an attained lower bound, independent of (a). Further if (a) is restricted to the class of non-constant monotonic sequences this lower bound cannot be attained, and it is natural to ask if it can be improved for such sequences.

The following theorem generalising one due to Čakalov answers this question, **[6, 9]**.

Theorem 7. *If $n>2$, (a) a positive sequence such that, $a_n>\max(a_1,\dots,a_{n-1})$ then*

(22) $$A_n(a;w)-G_n(a;w)\geqq\lambda_n\{A_{n-1}(a;w)-G_{n-1}(a;w)\},$$

where $\lambda_n=\inf_{1\leq j\leq n-1}\frac{W_{n-1}^2(W_n-w_j)}{W_n^2(W_{n-1}-w_j)}$, *with equality only when $a_1=\cdots=a_n$. Further if λ_n is replaced by a $\lambda_n'>\lambda_n$ there is a sequence (a) for which inequality (22) fails to hold.*

Proof. Write $\alpha_n = A_n(a;w) - G_n(a;w)$; then if $\varepsilon > 0$, $a_j = 1-\varepsilon$, $j \neq n$, $a_k = 1$, $k \neq j$,

$$\frac{\alpha_n}{\alpha_{n-1}} = \frac{\left(1-\frac{w_j}{W_n}\varepsilon\right)-(1-\varepsilon)^{\frac{w_j}{W_n}}}{\left(1-\frac{w_j}{W_{n-1}}\varepsilon\right)-(1-\varepsilon)^{\frac{w_j}{W_{n-1}}}},$$

and as ε decreases to zero $\frac{\alpha_n}{\alpha_{n-1}}$ decreases to λ_n for an appropriate choice of j. This proves the last part of the above statement.

Suppose now that $a_{n-p+1} = \cdots = a_n = x$, $a_1 \leqq a_2 \leqq \cdots \leqq a_{n-p} < x$ and write

$$g_r(x) = g(x) = W_n^2(W_{n-1}-w_1)\big(A_n(a;w) - G_n(a;w)\big) - W_{n-1}^2(W_n - w_1)\big(A_{n-1}(a;w) - G_{n-1}(a;w)\big)$$

simple calculations show that if $\Phi(t) = g'(x)$, where

$$t^{W_n W_{n-1}} = \big(G_{n-p}(a;w)\big)^{W_{n-p}} x^{-W_{n-p}},$$

then

$$\Phi'(t) = W_n W_{n-1} t^{W_{n-1}-1}\left\{(W_n - w_1)(W_{n-1} - W_{n-p}) t^{w_n} - (W_{n-1} - w_1)(W_n - W_{n-p})\right\},$$

and so if $0 < t < 1$, $\Phi'(t) \leqq 0$; since $\Phi(1) = 0$ this implies that if $x \geqq a_{n-p}$, $g'(x) > 0$. This clearly implies inequality (22) for any increasing sequence (a); since A_n, A_{n-1}, G_n, G_{n-1} are symmetric in $a_1, \ldots, a_{n-1}$ the result follows, and the cases are equality are immediate.

Remarks. (i) Since $\lambda_n > \frac{W_{n-1}}{W_n}$ inequality (22), is stronger than (16) with $s = 1$, $r = 0$.

(ii) It is not possible to improve inequality (19) in a similar manner. The result in [6] that purports to give such an improvement is false as was pointed out by DIANANDA. In fact if $a_1 = \cdots = a_{n-2} = 1$, $a_{n-1} = a_n = a$. Then $\lim_{a \to +\infty} (A_n/G_n) / (A_{n-1}/G_{n-1})^{K_n} = 0$ if $K_n > \frac{n-1}{n}$. In other words the exponent $\frac{n-1}{n}$ in (19) cannot be improved by assuming that $a_n \geqq \max(a_1, \ldots, a_{n-1})$.

(iii) Also, these inequalities cannot be extended to means with different weights, [6].

4. In [13] EVERITT raised a very interesting question connected with RADO's inequality; inequality (17) implies that $\lim_{n\to\infty} n(A_n - G_n)$ exists; what distinguishes the sequences for which this limit is finite? EVERITT divided the problem into four cases; (α) (a) is unbounded, (β) (a) is bounded but does not converge, (γ) (a) converges to zero, (δ) (a) converges to a finite, non-zero limit. In cases (α), (β), $\lim_{n\to\infty} n(A_n - G_n) = \infty$; in case ($\gamma$) $\lim_{n\to\infty} n(A_n - G_n) < \infty$ if and only if $\sum_{n\in N} a_n$ converges; in case (δ) $\lim_{n\to\infty} n(A_n - G_n) < \infty$ if and only if (a) is a strictly positive sequence such that for some finite positive a, $\sum_{n\in N}(a_n - a)^2 > \infty$. EVERITT's original proof of case (δ) contained a flaw. This was pointed out in a review by DIANANDA, [11], who later gave a correct proof [14]. The interesting part of the proof in [12], is that it makes essential use of the sub-additivity of the function μ defined in (b) following Theorem 5 with $(p) = (q)$, $r = 0$, $s = t = u = 1$.

5. The two basic inequalities that have given their name to inequalities of a similar type are inequalities (17) and (19). In recent literature, in particular in the standard work **[18]**, these have been called RADO's inequality and POPOVICIU's inequality respectively.

Inequality (17) appeared apparently for the first time in 1934 in the classical work of HARDY, LITTLEWOOD and PÓLYA, **[15,** Theorem 61] — where a proof entirely different to any given in this note is indicated. Since in this book the inequality is attributed to R. RADO the name has become standard. Various proofs have been given since then; besides those in the various papers mentioned in the bibliography the reader is referred to the excellent small book **[26]** of D. S. MITRINOVIĆ and P. M. VASIĆ, in particular to the papers mentioned on pages 31-32 and 34.

For the second inequality the situation is less standard. Certain authors, as for instance P. S. BULLEN **[7]** called this inequality POPOVICIU's inequality, since its proof was given in 1960 by T. POPOVICIU **[25]**. In book **[18]** this inequality is also called POPOVICIU's inequality. It seems, however, that this inequality was first proved in 1932 by F. SIMONART **[27]**. After this the inequality in question was rediscovered a number of times. Consult also the book **[26]** by D. S. MITRINOVIĆ and P. M. VASIĆ, particularly pp. 32-34.

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