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330. NOTES ON INEQUALITIES INVOLVING TRIANGLES OR TETRAHEDRONS

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NOTE I

TRIANGLE INEQUALITIES

Four inequalities of Oppenheim involving elements of two triangles are extended to elements of n triangles.

In this note, we give extensions to four triangle inequalities of OPPENHEIM which appeared in *Problem* 5092 (Amer. Math. Monthly, **71** (1964), 444—445). Our derivation is similar to the published solution (loc. cit.) by NOLAN.

Suppose that A_i, B_i, C_i $(i=0, 1, \ldots, n-1)$ are n triangles with sides a_i, b_i, c_i, area Δ_i, and altitudes p_i, q_i, r_i. If a_n, b_n, c_n are defined by the equations

$$a_n{}^2=\sum_0^{n-1} a_i^2, \quad b_n{}^2=\sum_0^{n-1} b_i^2, \quad c_n{}^2=\sum_0^{n-1} c_i^2,$$

then we will show that

(i) a_n, b_n, c_n are the sides of a triangle,

(ii) $p_n{}^2 \geqq \sum_0^{n-1} p_i^2, \quad q_n{}^2 \geqq \sum_0^{n-1} q_i^2, \quad r_n{}^2 \geqq \sum_0^{n-1} r_i^2,$

equality occurring in all three if and only if the original n triangles are similar,

(iii) $\Delta_n \geqq \sum_0^{n-1} \Delta_i$, with equality if and only if the original n triangles are similar,

(iv) $\Delta_n^n \geqq n^n \prod_0^{n-1} \Delta_i$, with equality if and only if the original n triangles are congruent.

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(i) follows immediately from the MINKOWSKI inequality

$$(x_1^m + x_2^m + \cdots + x_n^m)^{1/m} + (y_1^m + y_2^m + \cdots + y_n^m)^{1/m}$$
$$\geqq \{(x_1+y_1)^m + (x_2+y_2)^m + \cdots + (x_n+y_n)^m\}^{1/m}$$

where $x_i,\ y_i \geqq 0,\ m > 1$.

(ii) From the law of cosines, we get

$$c_n a_n \cos B_n = \sum_0^{n-1} c_i a_i \cos B_i .$$

Squaring and applying CAUCHY's inequality gives

$$c_n^2 \cos^2 B_n \leqq \left\{ \sum_0^{n-1} a_i^2/a_n^2 \right\} \left\{ \sum_0^{n-1} c_i^2 \cos^2 B_i \right\}.$$

Whence,

$$c_n^2 \sin^2 B_n \geqq \sum_0^{n-1} c_i^2 \sin^2 B_i$$

or that

$$p_n^2 \geqq \sum_0^{n-1} p_i^2 \text{ and similarly for } q_n^2 \text{ and } r_n^2.$$

Equality holds for p_n^2 if and only if

$$\frac{c_i \cos B_i}{a_i} = k \qquad (i = 0, 1, \ldots, n-1).$$

Therefore for equality for one altitude, similarity is sufficient but not necessary. Equality for two or three altitudes holds if and only if the original n triangles are similar.

$$\text{(iii)}\ 2\sum_0^{n-1} \Delta_i = \sum_0^{n-1} p_i a_i \leqq \left(\sum_0^{n-1} p_i^2 \right)^{\frac{1}{2}} \left(\sum_0^{n-1} a_i^2 \right)^{\frac{1}{2}} \leqq p_n a_n = 2\Delta_n .$$

Again, we have equality if and only if the original n triangles are similar.

(iv) By the arithmetic-geometric mean inequality,

$$\Delta_n \geqq \Delta_0 + \Delta_1 + \cdots + \Delta_{n-1} \geqq n(\Delta_0 \Delta_1 \cdots \Delta_{n-1})^{1/n}$$

or

$$\Delta_n^n \geqq n^n \prod_0^{n-1} \Delta_i$$

with equality if and only if the original n triangles are congruent.

It also follows from (i) that

$$\{\Sigma a_i^m\}^{1/m}, \quad \{\Sigma b_i^m\}^{1/m}, \quad \{\Sigma c_i^m\}^{1/m} \qquad (m > 1)$$

are the sides of a triangle.

Note II

INEQUALITIES FOR A TRIANGLE ASSOCIATED WITH n GIVEN TRIANGLES

Various inequalities concerning the elements of a triangle associated with n given triangles are derived. Two of these inequalities include, as special cases, two known inequalities for a pair of associated triangles.

1. Introduction

For any triangle ABC it is known [1, p. 12] that

(1) $$abc \geqq (a+b-c)(b+c-a)(c+a-b) \qquad \{E\}$$

where for convenience the symbol $\{E\}$ will denote "with equality if and only if triangle ABC is equilateral." A simple proof follows by noting that $a^2 \geqq a^2-(b-c)^2$, etc. By interpreting (1) geometrically, we are led to several generalizations by an averaging process over the sides of the triangle. Dually, we are led to other inequalities by an averaging process over the angles of the triangle.

2. Area Inequality for Two Related Triangles

We consider another triangle $A'B'C'$ where

$$a'=\frac{b+c}{2}, \quad b'=\frac{c+a}{2}, \quad c'=\frac{a+b}{2}.$$

Since $s=s'$ and $\triangle A'B'C'$ is "closer" to an equilateral triangle than $\triangle ABC$, we should expect that $\Delta' \geqq \Delta$ $\{E\}$. Since here, $\Delta'=(abcs)^{\frac{1}{2}}$, the latter inequality is equivalent to (1).

More generally, we should expect the same area inequality for any reasonable averaging transformation which makes $\triangle A'B'C'$ „more equilateral" than $\triangle ABC$. More precisely, if

$$a'=ua+vb+wc,$$

$$b'=va+wb+uc,$$

$$c'=wa+ub+vc,$$

where

$$u+v+w=1, \quad u, v, w \geqq 0,$$

then

(2) $$s'=s \text{ and } \Delta' \geqq \Delta \ \{E\}.$$

This triangle inequality is equivalent to

(3) $$(xa+yb+zc)(ya+zb+xc)(za+xb+yc) \geqq (a+b-c)(c+a-b)(b+c-a)$$

where

$$x+y+z=1, \quad -1 \leqq x, y, z \leqq 1$$

Expanding out and using

$$\Sigma a=2s, \ \Sigma ab=s^2+4Rr+r^2, \quad abc=4Rrs,$$

(3) can be rewritten as

(4) $$(1+xyz)(2s^3-6sr^2)+12Rrs(5xyz-\Sigma xy) \geqq (1-\Sigma x^2y)\Sigma a^2b+(1-\Sigma xy^2)\Sigma ab^2.$$

For the special case when $x-1=y=z=0$ which corresponds to (1), (4) reduces to the well known inequality $R\geqq 2r$.

A proof of (2) will follow from the next section.

3. Area Inequality for *n* Triangles

Let a_i, b_i, c_i denote the sides of the n triangles $A_iB_iC_i$ $(i=1, 2, \dots, n)$. Then the three numbers

$$a=\sum w_i a_i,\quad b=\sum w_i b_i,\quad c=\sum w_i c_i$$

where $\Sigma w_i=1$, $w_i\geqq 0$, are possible lengths of sides for a triangle ABC. Then,

$$s=\sum_i w_i s_i$$

and

$$\Delta^2=\sum_i w_i s_i \sum_i w_i(s_i-a_i)\sum_i w_i(s_i-b_i)\sum_i w_i(s_i-c_i).$$

Using CAUCHY's inequality twice,

(5) $$\sqrt{\Delta}\geqq \sum_i w_i\sqrt{\Delta_i} \qquad \{S_n\}$$

where the symbol $\{S_n\}$ denotes "with equality if and only if the n triangles are directly similar". Also, since

$$r^2s=(s-a)(s-b)(s-c) \quad\text{and}\quad 4R\Delta=abc,$$

we obtain by applying HÖLDER's inequality that

(6) $$(r^2s)^{1/3}\geqq \sum_i w_i(r_i^2s_i)^{1/3} \quad \{S_n\},$$

(7) $$(\Delta R)^{1/3}\geqq \sum_i w_i(\Delta_i R_i)^{1/3} \quad \{S_n\}.$$

If we now let

(8) $$n=3,\quad (a_2, b_2, c_2)=(b_1, c_1, a_1),\quad (a_3, b_3, c_3)=(c_1, a_1, b_1),$$

then (5) reduces to (2), (6) reduces to $r\geqq r_1$ and (7) reduces to $\Delta R\geqq \Delta_1 R_1$ or equivalently $Rr\geqq R_1 r_1$ (all $\{E\}$).

4. More Inequalities for *n* Triangles

We now consider an averaging process over the angles of the n triangles $A_iB_iC_i$. Let the angles of $\triangle ABC$ be given by

$$A=\sum_i w_i A_i,\quad B=\sum_i w_i B_i,\quad C=\sum_i w_i C_i$$

where the w_i's are weights as before. Since $\triangle ABC$ is again in some sense more equilateral than the set of n triangles $A_iB_iC_i$, we should expect some inequality pertaining to the isoperimetric ratio s^2/Δ. More precisely, we will show that

$$\frac{s^2}{\Delta} \leqq \sum_i w_i \frac{s_i^2}{\Delta_i} \quad \{S_n\}. \tag{9}$$

Since $\cot \theta/2$ is convex for $0 \leqq \theta \leqq \pi$ and

$$\frac{s^2}{\Delta} = \cot \sum_i w_i \frac{A_i}{2} + \cot \sum_i w_i \frac{B_i}{2} + \cot \sum_i w_i \frac{C_i}{2},$$

we get

$$\frac{s^2}{\Delta} \leqq \sum_i w_i \left\{ \cot \frac{A_i}{2} + \cot \frac{B_i}{2} + \cot \frac{C_i}{2} \right\}$$

which is equivalent to (9).

In a similar fashion, using the concavity of $\sin x$, we give extensions for the following known result [1, p. 90], [2, p. 326]:

If A', B', C' denote the second points of intersection of the angle-bisectors and the circumcircle of a triangle ABC, then

$$\text{area}\, A'B'C' \geqq \text{area}\, ABC \quad \{E\}. \tag{10}$$

5. Inequality Involving Circumradius and Inradius

$$\frac{r}{4R} = \sin \sum_i w_i \frac{A_i}{2} \sin \sum_i w_i \frac{B_i}{2} \sin \sum_i w_i \frac{C_i}{2} \geqq \sum_i w_i \sin \frac{A_i}{2} \sum_i w_i \sin \frac{B_i}{2} \sum_i w_i \sin \frac{C_i}{2}.$$

Then by HÖLDER's inequality,

$$\frac{r}{4R} \geqq \left\{ \sum_i w_i \left(\sin \frac{A_i}{2} \sin \frac{B_i}{2} \sin \frac{C_i}{2} \right)^{\frac{1}{3}} \right\}^3$$

or

$$\left\{ \frac{r}{R} \right\}^{\frac{1}{3}} \geqq \sum_i w_i \left\{ \frac{r_i}{R_i} \right\}^{\frac{1}{3}} \quad \{S_n\}. \tag{11}$$

Then trivially, $r/R \geqq \min_i (r_i/R_i)$.

6. Inequality Involving Semi-Perimeter and Circumradius

$$\frac{s}{R} = \sin \sum_i w_i A_i + \sin \sum_i w_i B_i + \sin \sum_i w_i C_i \geqq \sum_i w_i (\sin A_i + \sin B_i + \sin C_i)$$

or

$$\frac{s}{R} \geqq \sum_i w_i \frac{s_i}{R_i} \quad \{S_n\}. \tag{12}$$

Then trivially, $s/R \geqq \min_i (s_i/R_i)$.

7. Inequality Involving Area and Circumradius

Since $4\Delta R = abc$,

$$\frac{\Delta}{2R^2} = \sin \sum_i w_i A_i \sin \sum_i w_i B_i \sin \sum_i w_i C_i \geqq \{\sum_i w_i (\sin A_i \sin B_i \sin C_i)^{\frac{1}{3}}\}^3$$

or

$$(13) \qquad \left\{\frac{\Delta}{R^2}\right\}^{\frac{1}{3}} \geqq \sum_i w_i \left\{\frac{\Delta_i}{R_i^2}\right\}^{\frac{1}{3}} \qquad \{S_n\}.$$

Then trivially, $\Delta/R^2 \geqq \min_i (\Delta_i/R_i^2)$.

8. Special Cases

We now specialize inequalities (9), (11), (12) and (13) by subjecting them to conditions (8). These then become

$$(9)' \qquad \frac{\Delta}{s^2} \geqq \frac{\Delta_1}{s_1^2} \quad \{E\},$$

$$(11)' \qquad \frac{r}{R} \geqq \frac{r_1}{R_1} \quad \{E\},$$

$$(12)' \qquad \frac{s}{R} \geqq \frac{s_1}{R_1} \quad \{E\},$$

$$(13)' \qquad \frac{\Delta}{R^2} \geqq \frac{\Delta_1}{R_1^2} \quad \{E\}.$$

If additionally, $\triangle ABC$ is constrained to have the same circumcircle as $\triangle A_1B_1C_1$, then

$$(11)'' \qquad r \geqq r_1 \quad \{E\},$$

$$(12)'' \qquad s \geqq s_1 \quad \{E\},$$

$$(13)'' \qquad \Delta \geqq \Delta_1 \quad \{E\}.$$

It is to be noted that (10) is a special case of (13)″.

We conclude with some trigonometric versions of the latter inequalities by specializing conditions (8) still further, i.e., $w_1 = w_2 = 1/2$, $w_3 = 0$ and dropping all subscripts. Since,

$$\frac{2s^2}{\Delta} = \frac{(\sin A + \sin B + \sin C)^2}{\sin A \sin B \sin C},$$

(9)′ becomes

$$(9)'' \qquad \left\{\frac{r}{2R}\right\}^{\frac{1}{2}} = \left\{2 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right\}^{\frac{1}{2}} \leqq \frac{\cos\frac{A}{2}\sin\frac{A}{2} + \cos\frac{B}{2}\sin\frac{B}{2} + \cos\frac{C}{2}\sin\frac{C}{2}}{\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}}$$

or equivalently

$$(9)''' \qquad \cot\frac{\pi - A}{4} + \cot\frac{\pi - B}{4} + \cot\frac{\pi - C}{4} \leqq \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} \quad \{E\}.$$

Similarly, we obtain

$$(11)''' \qquad \sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}\geqq\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\quad\{E\},$$

$$(12)''' \qquad \cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\geqq\sin A+\sin B+\sin C\quad\{E\}.$$

In a subsequent paper, we will give a more direct derivation of the last three inequalities.

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NOTE III

INEQUALITIES INVOLVING THE ELEMENTS OF TWO TRIANGLES

An inequality involving the elements of two triangles is given. This inequality includes, as special cases, the ones of Barrow and Tomescu as well as other well known ones.

The inequality

$$(1) \qquad a'^2+b'^2+c'^2\geqq(-1)^{n+1}\{2a'b'\cos nC+2b'c'\cos nA+2c'a'\cos nB\}$$

(n-integral), relating to the elements of two triangles ABC and $A'B'C'$ follows immediately by expanding out

$$(2) \qquad \{a'+(-1)^n(b'\cos nC+c'\cos nB)\}^2+\{b'\sin nC-c'\sin nB\}^2\geqq 0.$$

There is equality, if and only if,

$$(3) \qquad \frac{\sin A'}{\sin nA}=\frac{\sin B'}{\sin nB}=\frac{\sin C'}{\sin nC}.$$

And (3) implies that

$$nA=m\pi+A',\quad nB=m\pi+B',\quad nC=m\pi+C'$$

if n $=3m+1$ and that

$$nA=(m+1)\pi-A',\quad nB=(m+1)\pi-B',\quad nC=(m+1)\pi-C'$$

if $n=3m+2$. For if $n=3m+1$, let $nA=m\pi+A_1$, etc., then

$$\frac{\sin A'}{\sin A_1}=\frac{\sin B'}{\sin B_1}=\frac{\sin C'}{\sin C_1}.$$

Since also $A_1+B_1+C_1=\pi$, $\triangle A'B'C'\sim\triangle A_1B_1C_1$ and the rest follows (and similarly for the case $n=3m+2$).

The case when $n=1$ had been established by BARROW [1, p. 24] and used by MORDELL [2] in his proof of the ERDÖS-MORDELL inequality. Equivalently, it can be rewritten in the form

(4) $$a'^2+b'^2+c'^2 \geqq \frac{a'b'}{ab}(a^2+b^2-c^2)+\frac{b'c'}{bc}(b^2+c^2-a^2)+\frac{c'a'}{ac}(c^2+a^2-b^2).$$

If also $a'=b'=c'$, then

$$a^3+b^3+c^3+5abc \geqq (a+b)(b+c)(c+a) \quad \{E\}.$$

The symbol $\{E\}$ is to mean "with equality if and only if the triangle is equilateral." Since $a^3+b^3+c^3 \geqq 3abc$, this is a stronger inequality then

$$8(a^3+b^3+c^3) \geqq 3(a+b)(b+c)(c+a), \quad [1, \text{p. } 12].$$

The case $n=2$ is equivalent to

(5) $$\frac{a^2}{a'}+\frac{b^2}{b'}+\frac{c^2}{c'} \leqq \frac{R^2(a'+b'+c')^2}{a'b'c'}$$

and corresponds to problem E 2221 proposed by TOMESCU [2]. The latter inequality is a rather "rich" one since it includes the following well known inequalities as special cases:

If $a'=b'=c'$, then

(5.1) $$a^2+b^2+c^2 \leqq 9R^2 \quad \{E\}, \quad [1, \text{p. } 52],$$

or equivalently

$$\sin^2 A+\sin^2 B+\sin^2 C \leqq \frac{9}{4} \quad \{E\}, \quad [1, \text{p. } 18].$$

Since,

$$\sum \sin^2 A = 2+2\cos A\cos B\cos C,$$

we also have

$$\cos A\cos B\cos C \leqq \frac{1}{8} \quad \{E\}, \quad [1, \text{p. } 25].$$

If $a'=a$, $b'=b$, $c'=c$, then

(5.2) $$R^2(a+b+c) \geqq abc \quad \{E\}$$

or

$$\sin A+\sin B+\sin C \geqq 4\sin A\sin B\sin C = \sin 2A+\sin 2B+\sin 2C \quad \{E\}, \ [1, \text{p. } 18].$$

Since also $abc=4Rrs$,

$$R \geqq 2r \quad \{E\}, \quad [1, \text{p. } 48].$$

If $a'=a^2$, $b'=b^2$, $c'=c^2$ ($\triangle ABC$ is acute), then

(5.3) $$R(a^2+b^2+c^2) \geqq \sqrt{3}\,abc \quad \{E\}$$

or

$$\sin^2 A+\sin^2 B+\sin^2 C \geqq 2\sqrt{3}\sin A\sin B\sin C$$
$$=\frac{\sqrt{3}}{2}(\sin 2A+\sin 2B+\sin 2C) \quad \{E\}.$$

Since also $abc=4R\Delta$,

$$a^2+b^2+c^2\geqq 4\sqrt{3}\Delta \quad \{E\}, \quad [\mathbf{1}, \text{p. } 42].$$

If $a'=b,\ b'=c,\ c'=a$, then

$$(5.4) \qquad R^2(a+b+c)^2\geqq a^3c+b^3a+c^3b \quad \{E\},$$

and similarly

$$R^2(a+b+c)^2\geqq ac^3+ba^3+cb^3 \quad \{E\}.$$

Then by adding,

$$2R^2\{\sum a\}^2\geqq\{\sum a\}\{\sum a^3\}-\sum a^4 \quad \{E\}.$$

Letting $a'=b'=c'$ $(n\neq 3m)$ in (1), we get

$$(6) \qquad \frac{3}{2}\geqq(-1)^{n+1}\{\cos nA+\cos nB+\cos nC\} \quad \{E\}.$$

Since

$$\sum\cos 2nA=4(-1)^n\cos nA\cos nB\cos nC-1,$$

$$\sum\cos(2n+1)A=4(-1)^n\sin\frac{2n+1}{2}A\sin\frac{2n+1}{2}B\sin\frac{2n+1}{2}C+1,$$

we also have

$$(6.1) \qquad 1\geqq 8(-1)^{n+1}\cos nA\cos nB\cos nC \quad (2n\neq 3m) \quad \{E\},$$

$$(6.2) \qquad 1\geqq 8(-1)^n\sin\frac{2n+1}{2}A\sin\frac{2n+1}{2}B\sin\frac{2n+1}{2}C \quad (2n+1\neq 3m) \quad \{E\}.$$

Additionally, it is easy to show that

$$(7) \qquad 1\geqq(-1)^m\cos 3mA\cos 3mB\cos 3mC,$$

$$(8) \qquad 1\geqq(-1)^m\sin\frac{6m+3}{2}A\sin\frac{6m+3}{2}B\sin\frac{6m+3}{2}C.$$

(1) becomes rather complicated if we express the trigonometric functions in terms of the sides for large n. Consequently, we conclude with the case $n=4$, i.e.,

$$(9) \qquad \frac{R^2(a'+b'+c')^2a^2b^2c^2}{a'b'c'}\geqq\sum\frac{[a^2(b^2+c^2-a^2)]^2}{a'}.$$

If $a'=b'=c'$, then

$$(9.1) \qquad 9R^2a^2b^2c^2=\frac{9a^4b^4c^4}{16\Delta^2}\geqq\sum[a^2(b^2+c^2-a^2)]^2 \quad \{E\}.$$

(9.1) is stronger than $3\sqrt{3}(abc)^2\geqq(4\Delta)^3$ [**1**, p. 46] since by CAUCHY's inequality

$$27a^4b^4c^4\geqq\sum[4\Delta a^2(b^2+c^2-a^2)]^2\sum 1\geqq[64\Delta^3]^2 \quad \{E\}.$$

If $a'=a,\ b'=b,\ c'=c$, then

(9.2) $$R^2(a+b+c)^2 abc \geqq \sum a^3(b^2+c^2-a^2)^2 \quad \{E\}.$$

If $a'=a^2,\ b'=b^2,\ c'=c^2$ ($\triangle ABC$ — acute), then

(9.3) $$R^2(a^2+b^2+c^2) \geqq \sum a^2(b^2+c^2-a^2)^2$$

or

$$\frac{(a^2+b^2+c^2)^2}{64\Delta^2} \geqq \cos^2 A+\cos^2 B+\cos^2 C \quad \{E\}.$$

The latter is stronger than (5.3) since [1, p. 24]

$$\sum \cos^2 A \geqq 3/4.$$

If $a'=a^2(b^2+c^2-a^2)$, etc., ($\triangle ABC$ — acute), then

(9.4) $$(abc)^2 \geqq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2) \quad \{E\}.$$

The inequality is also valid for non-acute triangles since then the r.h.s. becomes negative. This is a weaker inequality than [1, p. 12]

$$abc \geqq (a+b-c)(b+c-a)(c+a-b)$$

for

$$\prod (a+b-c)^2 \geqq \prod (a^2+b^2-c^2)$$

implies that

$$\frac{32\Delta^4}{(a+b+c)^2} \geqq a^2b^2c^2 \cos A \cos B \cos C$$

or

$$\frac{r^2}{2R^2} \geqq \cos A \cos B \cos C = \frac{r^2}{2R^2} - IH^2 \quad [1, p. 50]$$

and conversely. Also (9.4) is a weaker inequality than either

(10) $$3\sqrt{3}(abc)^2 \geqq (4\Delta)^3 \quad \{E\}, \quad [1, p. 46]$$

or

(11) $$(4\Delta)^6 \geqq 27(a^2+b^2-c^2)^2(b^2+c^2-a^2)^2(c^2+a^2-b^2)^2 \quad \{E\}, [1, p. 42].$$

For coupling (10) and (11), we obtain

$$27(abc)^4 \geqq (4\Delta)^6 \geqq 27(a^2+b^2-c^2)^2(b^2+c^2-a^2)^2(c^2+a^2-b^2)^2 \quad \{E\}.$$

In a subsequent paper, we will consider related trigonometric inequalities.

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Note IV

INEQUALITIES INVOLVING TWO TRIANGLES OR TETRAHEDRONS

An inequality of Živanović relating to the area of a triangle associated with a given equilateral triangle is extended in several different ways.

0. Introduction

In this note, we extend the following result of Živanović **[1, 2]** in several ways:

If P denotes any point within or on an equilateral triangle ABC and if A', B', C' denote points symmetrically situated to P with respect to the sides BC, CA, AB, then

$$\text{area } A'B'C' \leqq \text{area } ABC \tag{1}$$

with equality if and only if P is the centroid of ABC.

1. An affine equivalent

The first extension is rather simple. We start with an arbitrary triangle ABC and instead of taking mirror images of P across the sides, we "reflect" P along rays parallel to the respective medians as in Fig. 1. Inequality (1) is still valid. A proof follows immediately by affinely transforming ABC into an equilateral triangle (which always can be done). Since parallelism, ratio of areas and ratio of lengths of segments are preserved, the result is equivalent to that of Živanović.

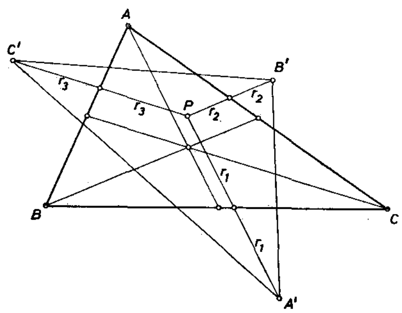

Fig. 1

2. Mirror reflections in an arbitrary triangle

We now show that (1) is also valid for arbitrary non-obtuse triangles. Here,

$$ar_1 + br_2 + cr_3 = 2\Delta,$$

$$\Delta' = 2[r_2 r_3 \sin A + r_3 r_1 \sin B + r_1 r_2 \sin C]$$

$$= 4\Delta\left[\frac{r_2 r_3}{bc} + \frac{r_3 r_1}{ca} + \frac{r_1 r_2}{ab}\right]$$

where Δ and Δ' denote the areas of triangles ABC and $A'B'C'$, respectively.

Changing to areal coordinates (except for a factor of 2)

$$x = ar_1, \qquad y = br_2, \qquad z = cr_3,$$

we wish to maximize

$$I = \frac{a^2b^2c^2}{4}\frac{\Delta'}{\Delta} = c^2xy + a^2yz + b^2zx$$

subject to the constraint

$$x + y + z = 2\Delta.$$

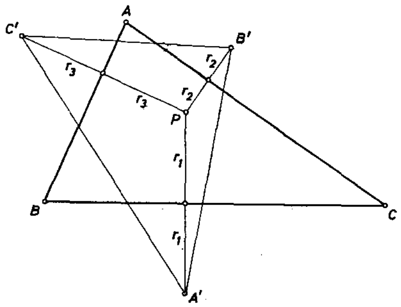

Fig. 2

At this stage, we first obtain the maximum by standard calculus techniques. Then from a knowledge of the result, we give a more elementary derivation by establishing the suggested extension of the well known inequality

$$\frac{x+y+z}{3} \geqq \left\{\frac{xy+yz+zx}{3}\right\}^{1/2}.$$

Eliminating z,

$$I = 2\Delta b^2x + 2\Delta a^2y + (c^2 - a^2 - b^2)xy - b^2x^2 - a^2y^2$$

subject to

$$x + y \leqq 2\Delta, \quad x, y \geqq 0.$$

For an interior point maximum, it is necessary that

$$\frac{\partial I}{\partial x} = 0 = 2\Delta b^2 + (c^2 - a^2 - b^2)y - 2b^2x,$$

$$\frac{\partial I}{\partial x} = 0 = 2\Delta y^2 + (c^2 - a^2 - b^2)x - 2a^2y.$$

Solving:

(2) $\quad 8\Delta x = a^2(b^2 + c^2 - a^2), \quad 8\Delta y = b^2(c^2 + a^2 - b^2), \quad 8\Delta z = c^2(a^2 + b^2 - c^2).$

Using the 2-nd derivative test, we show that the latter point corresponds to a maximum.

$$I_{xx} = -2b^2, \quad I_{yy} = -2a^2, \quad I_{xy} = c^2 - a^2 - b^2,$$

$$I_{xx}I_{yy} - I_{xy}{}^2 = 4a^2b^2 - (c^2 - a^2 - b^2)^2 = 16\Delta^2 > 0.$$

Corresponding to (2), we obtain

$$I_{\max} = \frac{a^2b^2c^2}{4}\frac{\Delta'}{\Delta} = \frac{a^2b^2c^2}{64\Delta^2}\sum(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)$$

or

$$\Delta'_{\max} = \Delta.$$

To complete the proof, we now check the endpoint extrema. For $x = 0$, $I = a^2y(2\Delta - y)$. Thus, $I_{\max} = a^2\Delta^2$. Similarly for $y = 0$, $I_{\max} = b^2\Delta^2$ and for $x + y = 2\Delta$ or $z = 0$, $I_{\max} = c^2\Delta^2$. Thus on the boundary of ABC,

$$\frac{\Delta'_{\max}}{\Delta} = \frac{4\Delta^2}{a^2b^2c^2}\max\{a^2, b^2, c^2\} = \max\{\sin^2 A, \sin^2 B, \sin^2 C\}.$$

For the case when one angle of ABC is a right angle, the maximizing point P occurs in the interior of the hypotenuse, i.e., if $C=\pi/2$, then $a^2+b^2=c^2$ and

$$P(x,y,z)=P(ab/2,\ ab/2,\ 0).$$

The previously established inequality suggests the following inequality

$$\frac{ap+bq+cr}{4\Delta}\geqq\left\{\frac{pq}{ab}+\frac{qr}{bc}+\frac{rp}{ca}\right\}^{1/2} \tag{3}$$

where a,b,c are sides of a non-obtuse triangle, $p,q,r\geqq 0$, and with equality if and only if

$$\frac{p}{a(b^2+c^2-a^2)}=\frac{q}{b(c^2+a^2-b^2)}=\frac{r}{c(a^2+b^2-c^2)}.$$

To establish (3) (which also implies (1)) in an elementary fashion, let

$$p=a(b^2+c^2-a^2)u,\quad q=b(c^2+a^2-b^2)v,\quad r=c(a^2+b^2-c^2)w.$$

(3) now follows since it can be rewritten as

$$\begin{aligned}&(b^2+c^2-a^2)(c^2+a^2-b^2)(8\Delta^2-a^2b^2)(u-v)^2\\&+(c^2+a^2-b^2)(a^2+b^2-c^2)(8\Delta^2-b^2c^2)(v-w)^2\\&+(a^2+b^2-c^2)(b^2+c^2-a^2)(8\Delta^2-c^2a^2)(w-u)^2\geqq 0\end{aligned}$$

(note that $16\Delta^2=\sum(2a^2b^2-a^4)$).

3. A perimeter inequality

Referring again to Figure 2 and assuming ABC is equilateral, we will show that

$$2s/\sqrt{3}\geqq s'\geqq s \tag{4}$$

(where as usual s denotes the semi-perimeter).
Since

$$B'C'^2=4r_2^2+4r_3^2-8r_2r_3\cos 2\pi/3,$$
$$s'=\sum\{r_2^2+r_2r_3+r_3^2\}^{1/2}.$$

Since also,

$$\{x^2+xy+y^2\}^{1/2}\leqq x+y \text{ (with equality if and only if } xy=0),$$

$$\{x^2+xy+y^2\}^{1/2}\geqq\frac{\sqrt{3}}{2}(x+y) \text{ (with equality if and only if } x=y),$$

we get

$$2(r_1+r_2+r_3)\geqq s'\geqq\sqrt{3}(r_1+r_2+r_3)$$

which is equivalent to (4). The l. h. s. equality occurs if and only if the point P is located at one of the vertices of ABC. The r. h. s. equality occurs if and only if P is located at the centroid of $\triangle ABC$.

4. Extension of (1) to tetrahedra

If P denotes any point within or on a regular tetrahedron $ABCD$ and if A', B', C', D' denote points symmetrically situated to P with respect to the faces BCD, CDA, DAB, ABC, then

$$\text{volume } A'B'C'D' \leqq \left(\frac{2}{3}\right)^3 \text{volume } ABCD \tag{5}$$

with equality if and only if P is the centroid of $ABCD$.

Assuming that $ABCD$ has edge length 2, its volume is given by $V(ABCD) = 2\sqrt{2}/3$.

Also

$$V(A'B'C'D') = V(PA'B'C') + V(PB'C'D') + V(PC'D'A') + V(PD'A'B').$$

We now use the volume formula for a tetrahedron as a function of the lengths of three coterminal edges and the angles between them [3]:

$$V = \frac{e_1 e_2 e_3}{6} \begin{vmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{12} & 1 & \cos\theta_{23} \\ \cos\theta_{13} & \cos\theta_{23} & 1 \end{vmatrix}^{1/2}.$$

If θ denotes a dihedral angle of $ABCD$, then $\cos\theta = 1/3$ and

$$\sphericalangle A'PB' = \sphericalangle B'PC' = \sphericalangle C'PA' = \pi - \theta.$$

Now letting

$$PA' = 2r_1, \quad PB' = 2r_2, \quad PC' = 2r_3, \quad PD' = 2r_4,$$

we obtain

$$V(PA'B'C') = \frac{4r_1r_2r_3}{3}\{1 - \cos^2\theta - 2\cos^3\theta\}^{1/2} = \frac{16\sqrt{3}\, r_1r_2r_3}{27}$$

and that

$$V(A'B'C'D') = \frac{16\sqrt{3}}{27}\{r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2\}.$$

The r_i's satisfy

$$r_1 + r_2 + r_3 + r_4 = \frac{3V}{F} = \frac{2\sqrt{6}}{3}$$

where F is the face area of $\triangle ABC$. Inequality (5) now follows immediately from the known inequality [4] for symmetric functions

$$r_1 + r_2 + r_3 + r_4 \geqq \left\{\frac{r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2}{4}\right\}^{1/3}$$

with equality if and only if $r_1 = r_2 = r_3 = r_4$ or equivalently that P is the centroid of $ABCD$.

If instead of reflecting P an equal distance across the faces, we reflect twice the distance across the faces (so that PA' is now $3r_1$ instead of $2r_1$, etc.), (5) becomes

$$V(A'B'C'D') \leq V(ABCD).$$

5. An affine equivalent for the tetrahedron

By an affine transformation **4.** extends analogously to **1.** (the affine equivalent of (1)).

6. A perimeter inequality for regular tetrahedrons

Here we obtain the 3-dimensional version of **3.** where we are reflecting *P* twice the distance across the faces. If *E* denotes the sum of the lengths of all the edges of *ABCD*, then

(6) $$E\sqrt{3/2} \geqq E' \geqq E.$$

The l. h. s. equality occurs if and only if *P* coincides with a vertex of *ABCD* whereas the r. h. s. equality occurs if and only if *P* is the centroid of *ABCD*.

Here,

$$A'B' = 3\left\{r_1^2 + \frac{2r_1r_2}{3} + \mathrm{r}_1^2\right\}^{1/2}$$

and thus

$$3(r_1 + r_2) \geqq A'B' \geqq \sqrt{6}\,(r_1 + r_2).$$

Whence,

$$3\sum(r_1 + r_2) \geqq E' \geqq \sqrt{6}\sum(r_1 + r_2)$$

or

$$9(r_1 + r_2 + r_3 + r_4) \geqq E' \geqq 3\sqrt{6}\,(r_1 + r_2 + r_3 + r_4)$$

which is equivalent to (6).

In a subsequent paper, we will give further extensions to simplices and also consider geometric properties other than length and content.

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