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A DIFFERENTIAL OPERATOR AND ITS APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS AND NONANALYTIC FUNCTIONS

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PREFACE

For partial differential equations one can pose the following problems:
PROBLEM 1. To determine the general solution of the given equation.
PROBLEM 2. To determine the particular solution of the given equation, which also satisfies some given additional conditions.

In the beginning the theory of partial differential equations was directed only to the solution of Problem 1. The expression

$$u(x, y)=f(x+ay)+g(x-ay), \tag{1}$$

obtained in 1747 by d'ALEMBERT and EULER is probably the first example of a general solution. It presents the general solution of the classical wave equation

$$a^2u_{xx}-u_{yy}=0. \tag{2}$$

where a is a constant.

This approach to partial differential equations dominated in the 18th and the 19th century and has led to important results. The theory of LAPLACE (communicated to the Academy of Sciences in Paris in 1771, and published in 1777) regarding the equation

$$u_{xy}+au_x+bu_y+cu+d=0,$$

where a, b, c, d are functions of x and y, and the more general theory of DARBOUX from 1870 are the most beautiful contibutions to that branch of mathematics.

However, partial differential equations appear not only in theoretical but also in practical problems of Physics and Engineering, which do not require general solutions, but rather those which also satisfy some additional conditions (boundary, initial, mixed), i.e., in practical problems one must solve Problem 2, and not Problem 1. In the theory of ordinary differential equations one can, as a rule, easily obtain the required particular solution starting with the general solution. D'ALEMBERT also found no difficulty in determining, starting with (1), the particular solution which satisfies the initial conditions

$$u(x, 0) = A(x), \quad u_y(x, 0) = B(x)$$

in the form

$$u(x, y) = \frac{1}{2}\big(A(x+ay) + A(x-ay)\big) + \frac{1}{2a}\int_{x-ay}^{x+ay} B(t)\, dt. \tag{3}$$

These two facts have led mathematicians of the 18th and 19th century to concentrate on solving Problem 1, hoping that the solution of Problem 2 can be easily obtained from the solution of Problem 1, as was done by d'ALEMBERT in the case of equation (2). This approach has only partially met with success. LAGRANGE succeeded in reducing Problem 1 for partial differential equations of first order to ordinary differential equations, but for partial differential equations of higher order we still do not possess general methods of integration. In fact, not only do we not know the general methods for determining the solution of Problem 1, but even if we know its solution we cannot, in general, use it to arrive at the solution of Problem 2. For example. though we know that the general solution of the LAPLACE equation

$$u_{xx} + u_{yy} = 0, \tag{4}$$

is given by

$$u(x, y) = f(x+iy) + g(x-iy), \tag{5}$$

still, using (5) we cannot solve, without further investigation, the main problems which depend on equation (4), as, for instance, the problem of electrical distribution.

For all those reasons in the second half of the 19th and in the 20th century the theory of partial differential equations has almost abandoned the general solutions. J. HADAMARD in his book [1] states that the equation by itself is undetermined until some supplementary conditions are added to it, when it becomes „determined". In the French Encyclopaedia [2] he says: ... *dans l'étude des équations aux dérivées partielles, plus encore que dans celle des équations différentielles ordinaires, on doit cesser de rechercher, comme le voulait l'Analyse classique, l'intégrale générale, c'est-à-dire une expression satisfaisant forcément à l'équation donnée E et susceptible, grâce aux éléments arbitraires qu'elle contient (constants ou fonctions), de représenter n'importe quelle solution de cette équation.* This opinion, given by such an authority, confirmed even more mathematicians in their disdain of general solutions.

However, though the „old" approach to partial differential equations was never completely abandoned (we mention, for example, DRACH's logic integration [3], [4], developed by G. HEILBRONN [5], [6]), only the newer times bring almost complete renaissance of the classic approaches to partial differential equations. So, for instance, referring to general solutions, which have been proclaimed useless for applied sciences, the applied mathematician W. F. AMES in his excellent book [7] says: *A knowledge of these general solutions is extremely important in the process of obtaining approximate solutions as well as acting* *as a guide to analytic methods.* He believes that the natural sciences will increase the interest in general solutions: *There is little mathematical interest in this area today, but hopefully the pressure from science and engineering will breathe new life with the subject...* Furthermore, AMES states on page 49 that in the theory of nonlinear partial differential equations the method of general solutions has proved more useful than the special methods, and later (page 180) he gives an example which shows how general solution can be used to generate the required particular solution, saying with some bitterness *The utility of this method received little recognition.*

Having in front of us those two, quite opposite opinions — the opinion of the great mathematician J. HADAMARD from 1923, which states that general solutions are not useful for practical problems, and a contemporary opinion from 1965 given by an applied mathematician that general solutions are often more useful, one once again comes to the old conclusion that abandoning a method *a priori* is absurd, that every approach has its place both in theory and applications, and that the difference between the so-called „classical" and „modern" mathematics is not so sharp.

This work is mainly devoted to the problems of finite integration of partial differential equations. It exploits an algebraic (in a way „modern") method for solving a („classical") problem of Analysis.

We introduce an operator A defined on a set of differentiable functions by the following axioms:

$$A(f_1+f_2)=Af_1+Af_2,$$

$$A(f_1 f_2)=f_1 Af_2+f_2 Af_1,$$

$$A\big(f_1(f_2)\big)=f_1' Af_2.$$

We then show that this operator is isomorphic with the ordinary derivative and we explain the transition from a solution of an ordinary differential equation to the corresponding solution of an equation which involves the operator A.

In the first chapter a special case of operator A, the expression $f(x, y)\dfrac{\partial}{\partial x}+g(x, y)\dfrac{\partial}{\partial y}$ is applied to partial differential equations, especially to linear equations of parabolic type. Application of operator A shows a complete analogy of a class of partial differential equations of parabolic type with ordinary linear differential equations, which at the same time explains why that class of parabolic equations can be reduced to ordinary differential equations. Moreover, we have shown (Chapter 2) how one can obtain, starting with the general solution of a partial differential equation of parabolic type (which has been obtained by the above method), the particular solution satisfying the given CAUCHY's initial conditions. In other words, it appears that general solutions are not always useless, since they yield the required particular solutions.

In the theory of nonanalytical functions (which we consider in the third Chapter) one interpretation of operator A has been introduced long time ago. Independently from each other, some special cases of operator A have been defined and examined by G. V. KOLOSOV, D. POMPEIU, E. R. HEDRICK.

I. N. VEKUA, A. BILIMOVIĆ and others. The general operator A serves however to give a better classification of nonanalytic functions, especially to bring out various analogies of that theory with the theory of analytic functions.

Using a special class of nonanalytic functions, called c-analytic functions we have generalised GOURSAT's theorem which states that real and imaginary parts of a complex function of the form $f(z)+\bar{z}\,g(z)$, where f and g are analytic functions, satisfy the equation $\Delta^2 u=0$, with $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. In fact, we have proved that real and imaginary parts of a complex function of the form

$$\sum_{k=0}^{n-1}\left(f_k(z)\,\bar{z}^k+g_k(\bar{z})\,z^k\right)$$

where $f_0, \ldots, f_{n-1}$ are analytic and $g_0, \ldots, g_{n-1}$ are c-analytic functions, satisfy the equation $\Delta^n u=0$. For another class of nonanalytic functions, also introduced here, we have proved a theorem analogous to CAUCHY's theorem on the integral over a closed curve. At the end of the third Chapter we give an interesting application of nonanalytic functions. Namely, in 1777 LAPLACE has proved that if the coefficients of the hyperbolic equation

$$u_{xy}+au_x+bu_y+cu=0 \tag{6}$$

satisfy one of the following conditions

$$a_x+ab-c=0 \quad \text{or} \quad b_y+ab-c=0$$

then one can obtain the general solution of (6) in a finite form. More than 100 years later, P. BURGATTI proved in 1895 the same result for elliptic equations

$$u_{xx}+u_{yy}+au_x+bu_y+cu=0. \tag{7}$$

whose coefficients satisfy

$$\frac{1}{2}a_x+\frac{1}{2}b_y+\frac{a^2+b^2}{4}-c=0 \quad \text{and} \quad a_y-b_x=0.$$

Applying the complex operator to (6), and then to (7), we see that the cited results are not independent, but rather that they follow from each other.

The fourth Chapter is devoted to systems of partial differential equations. Among other things, we have determined the solutions of the following systems

$$A(x, y)\,u_x-B(x, y)\,v_y=a_1(x, y)\,u-a_2(x, y)\,v+b_1(x, y),$$
$$A(x, y)\,v_x+B(x, y)\,u_y=a_2(x, y)\,u+a_1(x, y)\,v+b_2(x, y),$$

$$a_1u_x+a_2u_y-b_1v_x-b_2v_y=fu-gv+h_1,$$
$$b_1u_x+b_2u_y+a_1v_x+a_2v_y=gu+fv+h_2,$$

$$(p_1-1)\,u_x+p_2v_x-p_2u_y+(p_1+1)\,v_y=0,$$
$$p_2u_x-(p_1+1)\,v_x+(p_1-1)\,u_y+p_2v_y=0.$$

We have also obtained the solution of POLOŽIĬ's system

$$pu_x + qu_y - v_y = 0, \quad -qu_x + pu_y + v_x = 0,$$

if $\dfrac{p-1-iq}{p+1+iq}$ is an analytic function.

The solution of the famous system of VEKUA

$$u_x - v_y = au + bv + f, \quad v_x + u_y = cu + dv + g,$$

was determined if $a = d$, $b + c = 0$, or

$$B_{\bar z} B_z - B B_{\bar z z} - B^2 \bar{A}_z + B^2 A_z + B^3 \bar{B} = 0,$$

where

$$B = \frac{1}{4}[a - d + i(c+b)], \quad A = \frac{1}{4}[a + c + i(b-d)].$$

All the given solutions contain one arbitrary function and one arbitrary constant.

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CONTENTS

CONVENTIONS

1. In order to avoid repetition of phrases such as ,,arbitrary differentiable function" or ,,arbitrary twice differentiable function", etc., we have agreed to write „arbitrary function". From the very nature of the expressions which contain such functions, it can easily be seen what conditions od differentiability does the function have to satisfy.

2. When mixed derivatives u_{xy}, $w_{z\bar{z}}$, etc. were used, it was always supposed that, e.g., $u_{xy}=u_{yx}$.

3. We have used the letters $\mathcal{F}$ and $\mathcal{K}$ to denote the sets of all functions in two variables, and the set of complex functions. In fact, we have supposed that those functions are differentiable as many times as necessary.

0. INTRODUCTION

Let F be a set of differentiable functions, depending on one variable, and let F_1 be an other set such that $\{0, 1\} \subset F \cap F_1$. We agree to let 0, 1 denote respectively the functions $x \mapsto 0$, $x \mapsto 1$. Let A be a mapping of F into F_1, such that, for $f_1, f_2 \in F$, we have

$$(A1) \qquad A(f_1 + f_2) = Af_1 + Af_2,$$

$$(A2) \qquad Af_1 f_2 = f_1 Af_2 + f_2 Af_1,$$

$$(A3) \qquad Af_1(f_2) = f_1' \cdot Af_2.$$

We define the subset Φ of F by the following relation

$$(A4) \qquad \varphi \in \Phi \Leftrightarrow A\varphi = 0.$$

It can easily be shown that $\{0, 1\} \subset \Phi$. Indeed, by $(A1)$ we have

$$A0 = A(0+0) = A0 + A0 = 2A0,$$

i.e., $A0 = 0$, while, using $(A2)$ we obtain

$$A1 = A(1 \cdot 1) = 1A1 + 1A1 = 2A1,$$

i.e., $A1 = 0$, and hence according to $(A4)$, we see that $\{0, 1\} \subset \Phi$.

Supose that there exists at least one function $X \in F$, such that $AX = 1$. Clearly, then $A(X + \varphi) = 1$, for any $\varphi \in \Phi$.

Definition 0.1. *We shall say that the ordered quadruple (F, A, X, Φ) with the above properties represents a δ-system. The second component of that system will be called a δ-operator.*

Operator A_n is defined recursively:

$$A_1 f = Af$$

$$A_{n+1} f = A(A_n f) \qquad (n = 1, 2, \ldots).$$

Using $(A1)$—$(A3)$, we can prove the following formula

$$(0.1) \qquad Af(f_1, \ldots, f_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial f_i} Af_i.$$

Definition 0.2. *Relation of the form*

$$J(X, u, Au, \ldots, A_n u) = 0$$

is called the operator equation of the δ*-system* (F, A, X, Φ) *of order* n, *with respect to the unknown function* u.

Definition 0.3. *A solution of the operator equation is any function* u *which identically satisfies it.*

Fundamental theorem. *Let*

$$J(X, u, Au, \ldots, A_n u) = 0 \tag{0.2}$$

be an operator equation in the system (F, A, X, Φ), *and let*

$$J(Y, u, Bu, \ldots, B_n u) = 0 \tag{0.3}$$

be an operator equation of the system (G, B, Y, Ψ). *If* $u = f(X, \varphi_1, \ldots, \varphi_n)$ *is a solution of* (0.2), *then* $u = f(Y, \psi_1, \ldots, \psi_n)$ *is a solution of* (0.3). *Naturally,* $\varphi_1, \ldots, \varphi_n$ *are arbitrary elements of* Φ, *and* $\psi_1, \ldots, \psi_n$ *are arbitrary elements of* Ψ.

Proof. Using (0.1) we get

$$Af(X, \varphi_1, \ldots, \varphi_n) = f' AX = f', \quad A_2 f(X, \varphi_1, \ldots, \varphi_n) = Af' = f'' AX = f'',$$

and, furthermore,

$$A_n f(X, \varphi_1, \ldots, \varphi_n) = f^{(n)}. \tag{0.4}$$

Let $u = f(X, \varphi_1, \ldots, \varphi_n)$ be a solution of (0.2). Then

$$J(X, f, Af, \ldots, A_n f) = 0.$$

However, in virtue of (0.4) we have

$$J(X, f, f', \ldots, f^{(n)}) = 0. \tag{0.5}$$

But then

$$J(Y, f, f', \ldots, f^{(n)}) = 0, \tag{0.6}$$

since (0.5) and (0.6) are in fact the same identity, and hence

$$u = f(Y, \psi_1, \ldots, \psi_n)$$

is a solution of (0.3).

EXAMPLE 0.1. Let D be the set of all differentiable functions of one real variable x, and let C be the set of all real constants. Then $\left(D, \dfrac{d}{dx}, x, C\right)$ and $\left(D, x^2 \dfrac{d}{dx}, -\dfrac{1}{x}, C\right)$ are two δ-systems.

Since the general solution of the differential equation

$$y'' - 3y' + 2y = 0$$

(which we consider as an operator equation of the first system) is given by

$$y = C_1 e^x + C_2 e^{2x},$$

we conclude that the general solution of the differential equation

$$x^4y''+(2x^3-3x^2)y'+2y=0$$

(which we consider as the operator equation of the seccnd system) is given by

$$y=C_1 e^{(-1/x)}+C_2 e^{(-2/x)},$$

where in both cases C_1 and C_2 are arbitrary constants.

1. PARTIAL DIFFERENTIAL EQUATIONS

1.1. Basic concepts

In the theory of partial differential equations we can successfully use the following operator

$$f(x, y)\frac{\partial}{\partial x}+g(x, y)\frac{\partial}{\partial y}$$

where f and g are given functions of x, y. Clearly, $\left(\mathcal{G}, f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}, \omega, \Phi\right)$ is a δ-system, where the set Φ is described by the general solution of

$$f(x, y)\frac{\partial u}{\partial x}+g(x, y)\frac{\partial u}{\partial y}=0, \tag{1.1}$$

i.e., Φ is the set of all functions $F(\alpha)$, where $\alpha(x, y)$ is some solution of (1.1), and ω is any solution of

$$f(x, y)\frac{\partial u}{\partial x}+g(x, y)\frac{\partial u}{\partial y}=1.$$

We shall say that a solution of an n-th order partial differential equation is general if it contains n arbitrary functions.

1.2. First order partial differential equations

Let the functions α and ω be defined as above. Then the determination of the general solution of the equation

$$F\left(\omega(x, y),\ u,\ f(x, y)\frac{\partial u}{\partial x}+g(x, y)\frac{\partial u}{\partial y}\right)=0 \tag{1.2}$$

reduces to integration of the ordinary differential equation

$$F(x, y, y')=0. \tag{1.3}$$

In fact, if $G(x, y, C)=0$ is the general solution of (1.3), then $G\big(\omega(x, y), u, \alpha(x, y)\big)=0$ is the general solution of (1.2).

We shall give two examples of this method.

EXAMPLE 1.2.1. Equation

$$f(x, y)\frac{\partial u}{\partial x}+g(x, y)\frac{\partial u}{\partial y}=\alpha_1(x, y)u^2+\alpha_2(x, y)u+\alpha_3(x, y)$$

where α_1, α_2, α_3 are some particular solutions of (1.1), is analogous to RICCATI's equation

$$y=\alpha_1 y^2+\alpha_2 y+\alpha_3,$$

where α_1, α_2, α_3 are constants, and can, therefore, be integrated.

So, for example, we can integrate the following equation

$$y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=f_1(x^2-y^2)u^2+f_2(x^2-y^2)u+f_3(x^2-y^2),$$

where f_1, f_2, f_3 are arbitrary functions of the given arguments.

EXAMPLE 1.2.2. Equation

$$u=(\log x)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)+k\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right), \tag{1.4}$$

where k is an arbitrary function, is analogous to CLAIRAUT's equation

$$y=xy'+k(y'), \tag{1.5}$$

and since $y=Cx+k(C)$ is the general solution of (1.5), we have that

$$u(x, y)=f\left(\frac{x}{y}\right)\log x+k\left(f\left(\frac{x}{y}\right)\right)$$

is the general solution of (1.4). Besides, starting with the singular solution $y=f(x)$ of (1.5), we obtain the singular solution $u=f(\log x)$ of (1.4).

For instance, equation

$$u=\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\log x+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)^2$$

has the following general solution

$$u=f\left(\frac{x}{y}\right)\log x+\left[f\left(\frac{x}{y}\right)\right]^2,$$

and the following singular solution

$$u=-\frac{\log^2 x}{4}.$$

1.3. Second order equations of parabolic type

The operator equation

$$A_2 u+F(X)Au+G(X)u=0,$$

with

$$A=f(x, y)\frac{\partial}{\partial x}+g(x, y)\frac{\partial}{\partial y},$$

becomes

$$f^2\frac{\partial^2 u}{\partial x^2}+2fg\frac{\partial^2 u}{\partial x\,\partial y}+g^2\frac{\partial^2 u}{\partial y^2}+\Big(f\frac{\partial f}{\partial x}+g\,\frac{\partial f}{\partial y}+F(\omega(x,\,y))f\Big)\frac{\partial u}{\partial x}$$
$$+\Big(f\frac{\partial g}{\partial x}+g\,\frac{\partial g}{\partial y}+F(\omega(x,\,y)g)\frac{\partial u}{\partial y}+G(\omega(x,\,y))\,u=0. \tag{1.6}$$

Therefore, partial differential equation (1.6) reduces to the ordinary differential equation

$$y''+F(x)\,y'+G(x)\,y=0 \tag{1.7}$$

and if $C_1F_1(x)+C_2F_2(x)$ is the general solution of (1.7), (C_1, C_2 are arbitrary constants), then $C_1(\alpha)F_1(\omega)+C_2(\alpha)F_2(\omega)$ is the general solution of (1.6).

In the general theory of parabolic equations (see, for example, [8]) it is known that (1.6) can be reduced to (1.7), but the procedure is much longer.

The above result can be generalised. Namely, the equation

$$y''+F_1(x,\,C)\,y'+G_1(x,\,C)\,y=0$$

where C is a constant, is also a linear partial differential equation. The corresponding partial differential equation is

$$f^2\frac{\partial^2 u}{\partial x^2}+2fg\frac{\partial^2 u}{\partial x\,\partial y}+g^2\frac{\partial^2 u}{\partial y^2}+\Big(f\frac{\partial f}{\partial x}+g\,\frac{\partial f}{\partial y}+F_1(\omega,\,\alpha)f\Big)\frac{\partial u}{\partial x}$$
$$+\Big(f\frac{\partial g}{\partial x}+g\,\frac{\partial g}{\partial y}+F_1(\omega,\,\alpha)f\Big)\frac{\partial u}{\partial y}+G_1(\omega,\,\alpha)\,u=0. \tag{1.8}$$

However, if we suppose that α is not constant, e.g. that $\frac{\partial\alpha}{\partial x}\neq 0$, functions α and ω are independent. We can therefore always find two functions F_2, and G_2, such that

$$F_1(\omega,\,\alpha)=F_2(x,\,y)\quad\text{and}\quad G_1(\omega,\,\alpha)=G_2(x,\,y),$$

and equation (1.8) can be written in the following form

$$f^2\frac{\partial^2 u}{\partial x^2}+2fg\frac{\partial^2 u}{\partial x\,\partial y}+g^2\frac{\partial^2 u}{\partial y^2}+\Big(f\frac{\partial f}{\partial x}+g\frac{\partial f}{\partial y}+F_2(x,\,y)f\Big)\frac{\partial u}{\partial x}$$
$$+\Big(f\frac{\partial g}{\partial x}+g\,\frac{\partial g}{\partial y}+F_2(x,\,y)\,g\Big)\frac{\partial u}{\partial y}+G_2(x,\,y)\,u=0, \tag{1.9}$$

where F_2 and G_2 are arbitrary functions.

Equation (1.9) looks rather special. However, that is not true. In order to show that we shall start with an arbitrary parabolic equation

$$f^2\frac{\partial^2 u}{\partial x^2}+2fg\frac{\partial^2 u}{\partial x\,\partial y}+g^2\frac{\partial^2 u}{\partial y^2}+F(x,\,y)\frac{\partial u}{\partial x}+G(x,\,y)\frac{\partial u}{\partial y}+H(x,\,y)\,u=0. \tag{1.10}$$

Since F_2 which appears in (1.9) is arbitrary, we can put

$$F_2=\frac{F-f\dfrac{\partial f}{\partial x}-g\dfrac{\partial g}{\partial y}}{f}.$$

Equation (1.10) then becomes

$$f^2\frac{\partial^2 u}{\partial x^2}+2fg\frac{\partial^2 u}{\partial x\,\partial y}+g^2\frac{\partial^2 u}{\partial y^2}+\left(f\frac{\partial f}{\partial x}+g\,\frac{\partial f}{\partial y}+F_2 f\right)\frac{\partial u}{\partial x}+G(x,\,y)\frac{\partial u}{\partial y}+H(x,\,y)\,u=0.$$

It is well known that a condition for integrability of a parabolic equation

$$au_{xx}+2bu_{xy}+cu_{yy}+2du_x+2eu_y+fu=0, \qquad (b^2-ac=0)$$

i.e. the condition that it may be reduced to CHARPIT's system of partial differential equations, is given by

$$b\left(2d-\frac{\partial a}{\partial x}-\frac{b}{a}\frac{\partial a}{\partial y}\right)=a\left(2e-\frac{\partial b}{\partial x}-\frac{b}{a}\frac{\partial b}{\partial y}\right). \tag{1.11}$$

In our case we get

$$fg\,\left(Ff-f\,\frac{\partial f}{\partial x}-g\,\frac{\partial f}{\partial y}\right)=f^2\left(G-f\frac{\partial g}{\partial x}-g\,\frac{\partial f}{\partial x}-g\,\frac{\partial g}{\partial y}-\frac{g^2}{f}\frac{\partial f}{\partial y}\right),$$

i.e.,

$$G=f\frac{\partial g}{\partial x}+g\,\frac{\partial g}{\partial y}+Fg,$$

from where we see that equations (1.9) and (1.10), together with the condition (1.11) are equivalent.

This reduces integration of parabolic equation

$$au_{xx}+2bu_{xy}+cu_{yy}+2du_x+2eu_y+fu=0$$

with condition (1.11) to the problem of integration of ordinary linear differential equation of second order. Therefore, all the results in the theory of linear differential equations can be extended to such parabolic equations. So, for example, we have:

1° The general solution of equation (1.9) is of the from

$$u=C_1(\alpha)f_1(\omega)+C_2(\alpha)f_2(\omega)$$

where C_1, C_2 are arbitrary functions of α, and f_1 and f_2 are two particular solutions satisfying

$$\begin{vmatrix} f_1(\omega) & f_2(\omega) \\ \dfrac{df_1}{d\omega} & \dfrac{df_2}{d\omega} \end{vmatrix}\neq 0.$$

2° If one particular solution of (1.9) is known, then we can determine the general solution of the nonhomogeneous equation

$$f^{\,2}u_{xx}+2fgu_{xy}+g^2u_{yy}+(ff_x+gf_y+F_2 f)u_x+(fg_x+gg_y+F_2 g)\,u_y+G_2 u=H(x,\,y).$$

EXAMPLE 1.3.1. Equation

$$\frac{(x^2+y^2)^2}{16x^2y^2}(y^2u_{xx}+2xyu_{xy}+x^2u_{yy})-\frac{(x^2+y^2)(8x^4+x^2+y^2)}{16x^3}u_x \tag{1.12}$$

$$-\frac{(x^2+y^2)(8x^2y^2+x^2+y^2)}{16y^3}u_y+(x^2-y^2)u=0$$

satisfies the condition (1.11). Let us reduce it to the form (1.9). Dividing by $(x^2+y^2)^2$ we get

$$\frac{1}{16x^2}u_{xx}+\frac{1}{8xy}u_{xy}+\frac{1}{16y^2}u_{yy}-\frac{8x^4+x^2+y^2}{16x^3(x^2+y^2)}u_x-\frac{8x^2y^2+x^2+y^2}{16y^3(x^2+y^2)}u_y+\frac{x^2-y^2}{(x^2+y^2)^2}u=0.$$

It is now clear that (1.12) should be written as an operator equation in the system

$$\left(\mathcal{F},\ \frac{1}{4x}\frac{\partial}{\partial x}+\frac{1}{4y}\frac{\partial}{\partial y},\ x^2+y^2,\ \{\varphi(x^2-y^2)\}\right).$$

Equation (1.12) now becomes

$$\frac{1}{16x^2}u_{xx}+\frac{1}{8xy}u_{xy}+\frac{1}{16y^2}u_{yy}-\left(\frac{1}{16x^3}+\frac{x}{2(x^2+y^2)}\right)u_x-\left(\frac{1}{16y^3}+\frac{x^2}{2y(x^2+y^2)}\right)u_y+\frac{x^2-y^2}{(x^2+y^2)^2}u=0,$$

i.e.,

$$\frac{1}{16x^2}u_{xx}+\frac{1}{8xy}u_{xy}+\frac{1}{16y^2}u_{yy}+\left[\frac{1}{4x}\left(-\frac{1}{4x^2}\right)-\frac{1}{4x}\left(\frac{2x^2}{x^2+y^2}\right)\right]u_x \tag{1.13}$$

$$+\left[\frac{1}{4y}\left(-\frac{1}{4y^2}\right)-\frac{1}{4y}\left(\frac{2x^2}{x^2+y^2}\right)\right]u_y+\frac{x^2-y^2}{(x^2+y^2)^2}u=0.$$

Comparing (1.13) with (1.9), we see that

$$f(x,y)=\frac{1}{4x},\qquad g(x,y)=\frac{1}{4y},$$

$$F_2(x,y)=-\frac{2x^2}{x^2+y^2},\qquad G_2(x,y)=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

We must now express F_2, G_2 in the form

$$F_2(x,\ y)=F_1(x^2+y^2,\ x^2-y^2),\qquad G_2(x,\ y)=G_1(x^2+y^2,\ x^2-y^2).$$

Clearly,

$$F_1(u,v)=-\frac{u+v}{u},\qquad G_1(u,v)=\frac{v}{u^2}.$$

Therefore, the corresponding equation in the system $(\mathcal{F},\ A,\ X,\ \Phi)$ is

$$A_2u+F_1(X,\varphi)Au+G_1(X,\varphi)=0,\qquad \varphi\in\Phi,$$

or,

$$A_2u-\frac{X+\varphi}{X}Au+\frac{\varphi}{X^2}u=0,$$

or, in the system $\left(D, \frac{d}{dx}, x, C\right)$

$$y''-\frac{x+C}{x}y'+\frac{C}{x^2}y=0, \tag{1.14}$$

where C is a constant. A particular solution of (1.14) is $y_1=x^C e^x$, and so we can find its general solution $y=C_1 x^C e^x+C_2 y_2(x, C)$, where C_1, C_2 are arbitrary constants.

Hence, the general solution of (1.12) is

$$u=C_1(x^2-y^2)\, e^{x^2+y^2}(x^2+y^2)^{(x^2-y^2)}+C_2(x^2-y^2)\, y_2(x^2+y^2,\ x^2-y^2),$$

where C_1, C_2 are arbitrary differentiable functions of x^2-y^2.

EXAMPLE 1.3.2. Equation

$$\begin{aligned}\log^2(x+y)(y^2u_{xx}+2xyu_{xy}+x^2u_{yy})+\log(x+y)\left(x\log(x+y)+yf_1(x^2-y^2)\right)u_x\\ +\log(x+y)\left(y\log(x+y)+xf_1(x^2-y^2)\right)u_y+f_2(x^2-y^2)\,u=0.\end{aligned} \tag{1.15}$$

is analogous to EULER's equation

$$x^2y''+f_1xy'+f_2y=0, \tag{1.16}$$

where f_1, f_2 are constants. The general solution of (1.16) is given by $y=C_1x^\alpha+C_2x^\beta$, where α, β are different solutions of the equation $t^2+(f_1-1)\,t+f_2=0$. If $\alpha=\beta$, then $y=(C_1+C_2\log x)\,x$ presents the general solution of (1.16). Therefore, in the first case the general solution of (1.15) is given by

$$C_1(x^2-y^2)\,[\log(x+y)]^{\alpha(x^2-y^2)}+C_2(x^2-y^2)\,[\log(x+y)]^{\beta(x^2-y^2)},$$

where C_1, C_2 are arbitrary differentiable functions of x^2-y^2, and α and β, being the solutions of the equation

$$t^2+[f_1(x^2-y^2)-1]\,t+f_2(x^2-y^2)=0$$

also present functions of x^2-y^2.

If $\alpha=\beta$, the general solution of (1.15) is given by

$$[C_1(x^2-y^2)+C_2(x^2-y^2)\log\log(x+y)]\,[\log(x+y)]^{\alpha(x^2-y^2)}.$$

1.4. Nonlinear equations of second order

The exposed method of integration reduces, in the general case, any partial differential equation of the type

$$J\left(\omega,\ \alpha,\ f^2u_{xx}+2fgu_{xy}+g^2u_{yy}+\left(f\frac{\partial f}{\partial x}+g\frac{\partial f}{\partial y}\right)u_x+\left(f\frac{\partial g}{\partial x}+g\frac{\partial g}{\partial y}\right)u_y,\ fu_x+gu_y,\ u\right)=0$$

to the ordinary differential equation

$$J(x,\ C,\ y'',\ y',\ y)=0,$$

where C is a constant.

EXAMPLE 1.4.1. Equation

$$u\,(e^{2x}u_{xx}+2e^{x+y}u_{xy}+e^{2y}u_{yy})=\varphi(e^{-x}-e^{-y})\,(e^xu_x+e^yu_y)^2 \tag{1.17}$$

in the δ-system $(\mathcal{G}, A, X, \Phi)$ where

$$A=e^x\frac{\partial}{\partial x}+e^y\frac{\partial}{\partial y},\quad X=-e^{-x},\quad \Phi=\{\varphi(e^{-x}-e^{-y})\},$$

becomes

$$uA_2u = \varphi\,(Au)^2. \tag{1.18}$$

The general solution of (1.18) is

$$u = \begin{cases} |C_1X + C_2|^{\frac{1}{1-\varphi}} & \text{for } \varphi \neq 1, \\ C_1e^{C_2X} & \text{for } \varphi = 1, \end{cases}$$

where C_1, C_2 are arbitrary elements of Φ, and hence the general solution of (1.17) is

$$u\,(x, y) = \begin{cases} |C_1e^{-x} + C_2|^{\frac{1}{1-\varphi}} & \text{for } \varphi \neq 1 \\ C_1e^{C_2e^{-x}} & \text{for } \varphi = 1 \end{cases}$$

where C_1, C_2 are arbitrary differentiable functions of $e^{-x}-e^{-y}$.

1.5. A remark on equations of higher order

It is natural that the above method can be extended to partial differential equations of higher order. So, for instance, starting with the linear differential equation of third order

$$y''' + F(x)\,y'' + G(x)\,y' + H(x)\,y = 0,$$

we can solve the coresponding operator equation

$$A_3u + F(X)\,A_2u + G(X)\,Au + H(X)\,u = 0,$$

or, taking

$$A = f(x, y)\frac{\partial}{\partial x} + g\,(x, y)\frac{\partial}{\partial y},$$

the following partial differential equation

$$\begin{aligned} &f^3u_{xxx} + 3f^2gu_{xxy} + 3fg^2u_{xyy} + g^3u_{yyy} + \left(3f^2\frac{\partial f}{\partial x} + 3fg\frac{\partial f}{\partial y} + Ff^2\right)u_{xx} \\ &+ \left(3fg\frac{\partial f}{\partial x} + 3f^2\frac{\partial g}{\partial x} + 3g^2\frac{\partial f}{\partial y} + 3fg\frac{\partial g}{\partial y} + 2Ffg\right)u_{xy} + \left(3fg\frac{\partial g}{\partial x} + 3g^2\frac{\partial g}{\partial y} + Fg^2\right)u_{yy} \\ &+ \left(f^2\frac{\partial^2 f}{\partial x^2} + 2fg\frac{\partial^2 f}{\partial x\,\partial y} + g^2\frac{\partial^2 f}{\partial y^2} + f\left(\frac{\partial f}{\partial x}\right)^2 + g\frac{\partial f}{\partial y}\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} + g\frac{\partial g}{\partial y}\frac{\partial f}{\partial y}\right. \\ &\qquad \left. + Ff\frac{\partial f}{\partial x} + Fg\frac{\partial f}{\partial y} + Gf\right)u_x \\ &+ \left(f^2\frac{\partial^2 g}{\partial x^2} + 2fg\frac{\partial^2 g}{\partial x\,\partial y} + g^2\frac{\partial^2 g}{\partial y^2} + g\left(\frac{\partial g}{\partial y}\right)^2 + f\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial y}\frac{\partial g}{\partial x} + f\frac{\partial g}{\partial x}\frac{\partial g}{\partial y}\right. \\ &\qquad \left. + Ff\frac{\partial g}{\partial x} + Fg\frac{\partial g}{\partial y} + Gg\right)u_y + Hu = 0. \end{aligned}$$

For example, if $f(x, y)=x$, $g(x, y)=y$, we get

$$x^3u_{xxx}+3x^2yu_{xxy}+3xy^2u_{xyy}+y^3u_{yyy}+(3x^2+Fx^2)u_{xx}+(3xy+3xy+2Fxy)u_{xy}$$
$$+(3y^2+Fy^2)u_{yy}+(x+Fx+Gx)u_x+(y+Fy+Gy)u_y+Hu=0.$$

If F, G, H are constants, this equation becomes

$$\begin{aligned}x^3u_{xxx}+3x^2yu_{xxy}+3xy^2u_{xyy}+y^3u_{yyy}+ax^2u_{xx}+2axyu_{xy}\\+ay^2u_{yy}+bxu_x+byu_y+cu=0.\end{aligned}\tag{1.19}$$

Its general solution is

$$u(x, y)=\sum_{i=1}^{3} f_i\left(\frac{y}{x}\right)e^{t_i\log x}=\sum_{i=1}^{3} x^{t_i}f_i\left(\frac{y}{x}\right)\tag{1.20}$$

where f_1, f_2, f_3 are arbitrary differentiable functions of $\frac{y}{x}$, and $t_1\neq t_2\neq t_3\neq t_1$ are roots of the equation

$$t^3+(a-3)t^2+(b-a+2)t+c=0.$$

If F, G, H are functions of $\frac{y}{x}$, then (1.20) again presents the solution of (1.19), but in this case t_1, t_2, t_3 are also functions of $\frac{y}{x}$.

1.6. Further generalisations

The quoted method can be extended to partial differential equations which involve a function in several variables, $u(x_1, \ldots, x_n)$, since

$$A=\sum_{i=1}^{n} f_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i}$$

is also δ-operator.

2. CAUCHY'S PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS

Up to now we have only been concerned with determination of general solutions of partial differential equations which can be written in the form of an operator equation. We shall now show how CAUCHY's solution of such equations can also be obtained.

Let $J(x, y, y')=0$ be an ordinary differential equation, whose general solution is given by $A(x, y, C)=0$, where C is an arbitrary constant. Suppose that the equation

$$A(x_0, y_0, C)=0\tag{2.1}$$

where x_0, y_0 are given numbers can be solved with respect to C, i.e., that (2.1) implies

$$C = \Phi(x_0, y_0). \tag{2.2}$$

Then, CAUCHY's solution of the above equation is given by

$$A\big(x,\ y,\ \Phi(x_0, y_0)\big) = 0.$$

The equation which corresponds to the above equation in the system $(\mathcal{G}, A, X, \Phi)$ is

$$J\big(\omega(x, y),\ u,\ Au\big) = 0. \tag{2.3}$$

Its general solution is

$$A\Big(\omega(x, y),\ u,\ \varphi\big(\alpha(x, y)\big)\Big) = 0, \tag{2.4}$$

where φ is an arbitrary function. We shall look, however, for the solution of (2.4) which is such that when $y = f(x)$, $u = \Omega(x)$, i.e., such that

$$u\big(x,\ f(x)\big) = \Omega(x).$$

Let F be a function such that

$$\omega\big(x,\ f(x)\big) = F\Big(\alpha\big(x,\ f(x)\big)\Big).$$

If $\beta(x) = \alpha\big(x,\ f(x)\big)$ has its inverse function β^{-1}, then

$$F(t) = \omega\Big(\beta^{-1}(t),\ f\big(\beta^{-1}(t)\big)\Big).$$

Let us now determine the function Ω_1 such that

$$\Omega(x) = \Omega_1\Big(\alpha\big(x,\ f(x)\big)\Big).$$

We obtain

$$\Omega_1(t) = \Omega\big(\beta^{-1}(t)\big).$$

All we need is to solve with respect to $\varphi(\alpha)$ the following equation

$$A\big(F(\alpha),\ \Omega_1(\alpha),\ \varphi(\alpha)\big) = 0. \tag{2.5}$$

In virtue of (2.2), from (2.5) we get

$$\varphi(\alpha) = \Phi\big(F(\alpha),\ \Omega_1(\alpha)\big),$$

and the required CAUCHY's solution is given by

$$A\Big(\omega(x, y),\ u,\ \Phi\Big(F\big(\alpha(x, y)\big),\ \Omega_1\big(\alpha(x, y)\big)\Big)\Big) = 0.$$

EXAMPLE 2.1. The general solution of

$$xy' - y = 2x^2yy' \tag{2.6}$$

is given by

$$y = xy^2 + Cx. \tag{2.7}$$

CAUCHY's integral, i.e., the solution satisfying $y_0=y(x_0)$ where x_0, y_0 are given numbers, can be obtained if we put

$$C=\frac{y_0-x_0y_0^2}{x_0}.$$

Consider the equation

$$\log x\,(2u\log x-1)\,(xu_x-yu_y)+u=0. \tag{2.8}$$

Its solution with an arbitrary function is

$$u=\log x\,[u^2+f(xy)].$$

We require to find that solution of (2.8) which contains the curve $y=x^2$, $u=x^3$. Applying the above procedure, we get

$$\log x=F(x^3),\ \text{ i.e., }\ F(t)=\frac{1}{3}\log t,$$

$$x^3=\Omega_1(x^3),\ \text{ i.e., }\ \Omega_1(t)=t,$$

and therefore

$$f(\alpha)=\frac{\alpha-\frac{1}{3}\alpha^2\log\alpha}{\frac{1}{3}\log\alpha},$$

which means that the required solution is given by

$$u(x,y)=\log x\left[u^2+\frac{3xy-x^2y^2\log xy}{\log xy}\right].$$

EXAMPLE 2.2. The general solution of CLAIRAUT's equation

$$y=xy'-y'^2 \tag{2.9}$$

is

$$y=Cx-C^2,$$

where C is an arbitrary constant. In this case CAUCHY's problem is not correctly set, since if $x_0^2\neq 4y_0$, we have two integral curves passing through each point (x_0, y_0). Those two curves are obtained if we put

$$C=\frac{x_0\pm\sqrt{x_0^2-4y_0}}{2}. \tag{2.10}$$

We require to find those integral surfaces of the equation

$$u=\frac{1}{y}(x^2u_x-y^2u_y)-(x^2u_x-y^2u_y)^2 \tag{2.11}$$

which contain the curve $y=x$, $u=-\dfrac{15}{4x^2}$.

The general solution of equation (2.11) is given by

$$u(x,y)=\frac{1}{y}f\left(\frac{1}{x}+\frac{1}{y}\right)-\left[f\left(\frac{1}{x}+\frac{1}{y}\right)\right]^2.$$

Applying the above procedure, we have

$$\frac{1}{x}=F\left(\frac{2}{x}\right), \text{ i.e., } F(t)=\frac{t}{2}$$

$$-\frac{15}{4x^2}=\Omega_1\left(\frac{2}{x}\right), \text{ .e., } \Omega_1(t)=-\frac{15}{16}t^2.$$

Therefore, in virtue of (2.10),

$$f(\alpha)=\frac{\frac{\alpha}{2}\pm\sqrt{\frac{\alpha^2}{4}+\frac{15\alpha^2}{4}}}{2},$$

i.e.,

$$f(\alpha)=\frac{5\alpha}{4}, \quad f(\alpha)=-\frac{3\alpha}{2}.$$

Therefore, the following surfaces

$$u(x, y)=\frac{5}{4}\left(\frac{1}{x}+\frac{1}{y}\right)\frac{1}{y}-\frac{25}{16}\left(\frac{1}{x}+\frac{1}{y}\right)^2, \quad u(x, y)=-\frac{3}{4}\left(\frac{1}{x}+\frac{1}{y}\right)\frac{1}{y}-\frac{9}{16}\left(\frac{1}{x}+\frac{1}{y}\right)^2$$

satisfy equation (2.11) and contain the curve $y=x,\ u=-\dfrac{15}{4x^2}$.

A similar method can be applied to equations of higher order. The case of linear partial differential equations of parabolic type has been discussed in detail in **[9]**. We shall only cite the example which was given in that paper.

Example 2.3. The function u which satisfies the Bertrand equation

$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}+xu_x+yu_y=n^2u$$

(n is a constant) and is such that

$$u(x, x^2)=x^3, \qquad u_y(x, x^2)=x^2,$$

is given by

$$u(x, y)=\frac{1}{2n}\left(\left[(n+3)\frac{x}{y}-1\right]\left(\frac{x}{y}\right)^{n-4}x^n+\left[(n-3)\frac{x}{y}+1\right]\left(\frac{x}{y}\right)^{-n-4}x^{-n}\right).$$

3. COMPLEX OPERATORS. NONANALYTIC FUNCTIONS

3.1. Kolosov's operator D

On the set of complex functions w,

$$w(z)=u(x, y)+iv(x, y), \qquad z=x+iy,$$

where u and v are differentiable functions, G. V. Kolosov **[10]**, **[11]**, **[12]**, introduced the operator D defined by

$$Dw=\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

and used it to integrate various systems of partial differential equations which appear in Mathematical Physics, especially in the Theory of Elasticity. KOLOSOV first proved in [10] certain formulas which enable him to work with D as if it were a derivative, and then applied them in [12] using as an inspiration the corresponding ordinary differential equations. Later on he showed the general method of integration, which corresponds to our Fundamental Theorem.

KOLOSOV's operator D was rediscovered a number of times. The history of this problem, with an extensive literature, is given in [13] and [14].

In fact, it is not difficult to show that D is a δ-operator. The corresponding δ-system is $(\mathcal{K}, D, \bar{z}/2, \{f(z)\})$, where f are arbitrary analytic functions.

We shall now use an operator analogous to D to obtain an interesting property of a class of complex functions.

3.2. Generalisation of Goursat's theorem

As a generalisation of the known fact that the real and imaginary parts of an analytic function $f(z)=u(x, y)+iv(x, y)$ satisfy LAPLACE's equation $\Delta u=0$, $\Delta v=0$, we have the following GOURSAT's theorem (see, for example, [15], pp. 193—194).

The real and imaginary parts of a bianalytic functions, i.e., a function of the form

$$f_0(z)+\bar{z}f_1(z),$$

where f_0 and f_1 are analytic functions, satisfy MAXWELL's equation $\Delta^2 u=0$, $\Delta^2 v=0$.

Using the operator D, S. FEMPL [16] has generalised GOURSAT's theorem. He has proved that the real and imaginary parts of an „areolare polynomial", i.e., of the function

$$\sum_{k=0}^{n-1} f_k(z)\bar{z}^k$$

where f_k are analytic functions, satisfy the equation $\Delta^n u=0$, $\Delta^n v=0$. FEMPL has also proved that the areolare polynomials are the only functions whose n-th deviation from analyticity is an analytic function. The n-th deviation from analyticity is defined as $D_n w$, where $D_2=D(D)$, etc.

In paper [17] we have introduced c-analytic functions as complex functions whose real and imaginary parts satisfy the following system of partial differential equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0.$$

c-analytic functions are in connection with the operator $\bar{D}$ (also introduced by KOLOSOV):

$$\bar{D}w=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right).$$

Clearly, $\overline{D}$ is also a δ-operator, with the corresponding δ-system $(\mathcal{K}, \overline{D}, z/2, \{f(\overline{z})\})$, where $f(\overline{z})$ are c-analytic function.

We shall say that $D_n w$ is the n-th deviation from c-analyticity $(D_n = D(D_{n-1}), \ldots)$.

Notice that the statement „deviation of a function w from analyticity is an analytic function" is equivalent to „$D_2 w = 0$". Similarly, statements „deviation of a function w from c-analyticity is a c-analytic function" and „$\overline{D}_2 w = 0$" are equivalent.

Theorem 3.1. *Real and imaginary parts, u and v, of a function of the form*

$$f_0(\overline{z}) + z f_1(\overline{z}) \tag{3.1}$$

where f_0 and f_1 are c-analytic functions, satisfy Laplace's equation, i.e., we have

$$\Delta^2 u = 0, \quad \Delta^2 v = 0.$$

Moreover, functions of the form (3.1) *are the only functions whose deviation from c-analyticity is a c-analytic function.*

Proof.[1] We immediately see that for a function w of the form (3.1) we have

$$\overline{D}_2 w = 0. \tag{3.2}$$

Let us now prove the other parts of the theorem.

Put

$$\overline{D} w = U(x, y) + iV(x, y).$$

Then, condition (3.2) implies

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0,$$

and, as a consequence,

$$\Delta U = 0, \quad \Delta V = 0. \tag{3.3}$$

Since

$$U = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad V = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

we have

$$\frac{\partial}{\partial x}(\Delta U) - \frac{\partial}{\partial y}(\Delta V) = \Delta^2 u$$

and by (3.3) we obtain

$$\Delta^2 u = 0 \tag{3.4}$$

Similary, $\Delta^2 v = 0$.

Let us now prove that functions of the form (3.1) are the only such functions for which (3.2) holds.

[1] This is a correction of the proof given in **[17]**.

Equation (3.2) separates into the following system

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=-2\frac{\partial^2 v}{\partial x\,\partial y} \tag{3.5}$$

$$\frac{\partial^2 v}{\partial y^2}-\frac{\partial^2 v}{\partial x^2}=-2\frac{\partial^2 u}{\partial x\,\partial y}. \tag{3.6}$$

Start with equation (3.4) whose expanded form is

$$\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}=0.$$

The general solution of this equation is (see **[18]**)

$$u(x,\,y)=f(x+iy)+yg(x+iy)+\varphi(x-iy)+y\psi(x-iy), \tag{3.7}$$

where f, g, φ, ψ are arbitrary differentiable functions.

From (3.5) and (3.7) we get

$$\frac{\partial^2 v}{\partial x\,\partial y}=-f''(x+iy)+ig'(x+iy)-yg''(x+iy)-\varphi''(x-iy)$$
$$-i\psi'(x-iy)-y\psi''(x-iy),$$

and, integrating with respect to x,

$$\frac{\partial v}{\partial y}=-f'(x+iy)+ig(x+iy)-yg'(x+iy)-\varphi'(x-iy)$$
$$-i\psi(x-iy)-y\psi'(x-iy)+F(y),$$

where F is an arbitrary function.

Integrate now the obtained equation with respect to y. We find

$$v(x,\,y)=if(x+iy)+iyg(x+iy)-i\varphi(x-iy)-iy\psi(x-iy)+\Psi(y)+\Phi(x), \tag{3.8}$$

where Φ is an arbitrary differentiable function, and $\Psi(y)=\int F(y)\,dy$.

In order to determine Φ and Ψ we shall use, besides (3.8), equations (3.6) and (3.7). The last one becomes

$$\Psi''(y)-\Phi''(x)=0.$$

This means that

$$\Phi(x)=ax^2+bx+c,\quad \Psi(y)=ay^2+dy+e,$$

where a, b, c, d, e are arbitrary constants.

Therefore,

$$\Phi(x)+\Psi(y)=ax^2+bx+c+ay^2+dy+e,$$

i.e.,

$$\Phi(x)+\Psi(y)=\left(\frac{b\bar{z}}{2}+\frac{i}{2}d\bar{z}+c+e\right)+z\left(a\bar{z}+\frac{b}{2}-\frac{id}{2}\right)=\alpha(\bar{z})+z\beta(\bar{z}).$$

Function w now becomes

$$w(z)=u(x, y)+iv(x, y)=2\varphi(x-iy)+2y\psi(x-iy)+i\alpha(\bar{z})+iz\beta(\bar{z})$$
$$=2\varphi(\bar{z})-iz\psi(\bar{z})+i\alpha(\bar{z})+iz\beta(\bar{z})=f_0(\bar{z})+zf_1(\bar{z}),$$

where $f_0(\bar{z})=2\varphi(\bar{z})+i\bar{z}\psi(\bar{z})+i\alpha(\bar{z})$, $f_1(\bar{z})=-i\psi(\bar{z})+i\beta(\bar{z})$.

This completes the proof of the above theorem.

Continuing this procedure, we can show that a function of the form

$$f_0(\bar{z})+zf_1(\bar{z})+z^2f_2(\bar{z})$$

is such that its second deviation from c-analyticity is a c-analytic function, and that its real and imaginary parts satisfy

$$\Delta^3 u=0, \ \Delta^3 v=0.$$

By the use of an operator which is inverse to $\bar{D}$, we can prove a more general result, which is analogous to FEMPL's result on areolare polynomials, i.e., that a function of the form

$$\sum_{i=0}^{n} z^i f_i(\bar{z})$$

is such that its n-th deviation from c-analyticity is a c-analytic function, while its real and imaginary parts, u and v, satisfy the equations

$$\Delta^{n+1}u=0, \ \Delta^{n+1}v=0. \tag{3.9}$$

We shall give the most general result related to equations (3.9), which contains all the previous results.

Lemma 3.1. *Let $w(z, \bar{z})=u(x, y)+iv(x, y)$ be a complex function whose partial derivatives with respect to z and $\bar{z}$ are continuous. Then*

$$\frac{\partial^{2n} w}{\partial z^n \partial \bar{z}^n}=\frac{1}{2^{2n}}(\Delta^n u+i\Delta^n v). \tag{3.10}$$

Proof. Let $n=1$. Then

$$\frac{\partial^2 w}{\partial z \partial \bar{z}}=\frac{1}{4}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\left(\frac{\partial w}{\partial x}-\frac{\partial w}{\partial y}\right)=\frac{1}{4}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{1}{4}(\Delta u+i\Delta v).$$

Suppose that (3.10) holds for some n. Then

$$\frac{\partial^{2n+2} w}{\partial z^{n+1}\partial \bar{z}^{n+1}}=\frac{1}{4}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\frac{1}{2^{2n}}(\Delta^n u+i\Delta^n v)$$
$$=\frac{1}{2^{2n+2}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\Delta^n u+i\Delta^n v)$$
$$=\frac{1}{2^{2n+2}}\Delta(\Delta^n u+i\Delta^n v)=\frac{1}{2^{2n+2}}(\Delta^{n+1}u+i\Delta^{n+1}v).$$

This completes the induction proof.

Corollary.

$$\frac{\partial^{2n} w}{\partial z^n \partial \bar{z}^n}=0 \text{ if and only if } \Delta^n u=0 \text{ and } \Delta^n v=0.$$

Theorem 3.2. *Complex functions of the form*

$$\sum_{i=0}^{n-1} [\bar{z}^i f_i(z)+z^i g_i(\bar{z})], \tag{3.11}$$

where f_i, g_i $(i=0, \dots, n-1)$ *are analytic, c-analytic functions, respectively, and only those functions, have the property that their real and imaginary parts,* u *and* v, *satisfy*

$$\Delta^n u=0, \quad \Delta^n v=0. \tag{3.12}$$

Proof. Conditions (3.12) are equivalent to

$$\frac{\partial^{2n} w}{\partial \bar{z}^n \partial z^n}=0. \tag{3.13}$$

However, the general solution of (3.13) is (3.11).

3.3. A generalisation of Kolosov's operators

Kolosov's operator D can be generalised as follows: Define an operator K on the set of complex functions $w=u(x, y)+iv(x, y)$ by

$$Kw=A\frac{\partial u}{\partial x}-B\frac{\partial v}{\partial y}+i\left(A\frac{\partial v}{\partial x}+B\frac{\partial u}{\partial y}\right)$$

where A and B are functions of x, y.

Clearly K is a δ-operator.

According to the general theory, if f is any solution of the equation

$$Kw=1,$$

and g a solution of

$$Kw=0,$$

then equation

$$J(f, g, w, Kw, \dots, K_n w)=0 \tag{3.14}$$

in the system $(\mathcal{K}, K, f, \{g\})$ corresponds to equation

$$J(x, C, y, y', \dots, y^{(n)})=0, \tag{3.15}$$

in the system $\left(D, \frac{d}{dx}, x, C\right)$, where C is constant, and, therefore, integration of equation (3.14) reduces to integration of (3.15).

3.4. On nonanalytic functions

Nonanalytic functions w can be described by $Dw \neq 0$. Introducing some other special conditions, one can define subclasses of the class of nonanalytic functions. So, for example, systems

$$\sigma_1(x)\frac{\partial u}{\partial x}=\tau_1(y)\frac{\partial v}{\partial y},\quad \sigma_2(x)\frac{\partial u}{\partial y}=-\tau_2(y)\frac{\partial v}{\partial x},$$

$$\frac{\partial u}{\partial x}=\frac{1}{p}\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{1}{p}\frac{\partial v}{\partial x}$$

$$p\frac{\partial u}{\partial x}+q\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=0,\quad -q\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0,$$

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=au+bv+f,\quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=cu+dv+g,$$

where p, q, a, b, c, d, f, g are functions of x, y subjected to certain conditions define the so called Σ-monogenic, p-analytic, p, q-analytic, and generalised analytic functions, introduced by L. Bers and A. Gelbart **[19]** and **[20]** and G. N. Položiĭ **[21]** and I. N. Vekua **[22]**. Their complex forms are

$$(\sigma_1+\sigma_2+\tau_1+\tau_2)\frac{\partial w}{\partial \bar z}+(\sigma_1-\sigma_2+\tau_2-\tau_1)\frac{\partial w}{\partial z}+(\sigma_1+\sigma_2-\tau_2-\tau_1)\frac{\partial \bar w}{\partial \bar z}$$

$$+(\sigma_1-\sigma_2-\tau_2+\tau_1)\frac{\partial \bar w}{\partial z}=0,$$

$$\frac{\partial w}{\partial \bar z}=\frac{1-p}{1+p}\frac{\partial \bar w}{\partial \bar z},$$

$$(p+1-iq)\frac{\partial w}{\partial \bar z}+(p-1+iq)\frac{\partial \bar w}{\partial \bar z}=0,$$

$$\frac{\partial w}{\partial \bar z}=Aw+B\bar w+F,$$

$$A=\frac{1}{4}(a+d+ic-ib),\quad B=\frac{1}{4}(a-d+ic+ib),\quad F=\frac{1}{2}(f+ig).$$

More details about nonanalytic functions can be found in papers **[13]** and **[14]**.

It is not difficult to show that $D=2\frac{\partial}{\partial \bar z}$, $\bar D=2\frac{\partial}{\partial z}$. It is often stated that one can operate with $\frac{\partial}{\partial \bar z}$, $\frac{\partial}{\partial z}$ as if they were derivatives, and in some places even the authors prove some of their basic properties. In fact, $\frac{\partial}{\partial \bar z}$ and $\frac{\partial}{\partial z}$ are partial derivatives with respect to $\bar z$ and z, which can be considered as independent variables. In order to justify this assertion, we cite the following facts:

1° For two real functions u, v: $R^2 \to R$ it is said that they are independent if there does not exist a differentiable function φ such that $u=\varphi(v)$.

It is well known that the necessary and sufficient condition for independence of functions u and v is given by

$$J=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \neq 0.$$

Introducing the corresponding definition of independent functions u, v: $R^2 \to C$, we see that the condition of independence remains unchanged.

Consider now the functions

$$u(x, y)=x+iy, \quad v(x, y)=x-iy.$$

We have

$$J=\begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = -2i \neq 0,$$

and according to the above definition they are independent.

2° From RIEMANN's formula for the complex derivative (see [23])

$$\frac{dw}{dz}=\frac{1}{2}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]+\frac{1}{2}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]e^{-2i\varphi}$$

where $dz=\varepsilon e^{i\varphi}$, multiplying by dz, we obtain the expression which is analogous to the complete differential of a real function of two variables, namely (see [24])

$$dw=\frac{\partial w}{\partial z}\,dz+\frac{\partial w}{\partial \bar{z}}\,d\bar{z}.$$

This way of thinking enables us to classify complex functions in the following way:

A complex function, which has derivatives with respect to $\bar{z}$ and z, can be considered as a solution of a partial differential equation.

So for example analytic functions present the solution of the simplest partial differential equation $\frac{\partial w}{\partial \bar{z}}=0$. Another class of complex functions, as simple as the class of analytic functions, is the class of c-analytic functions, defined by $\frac{\partial w}{\partial z}=0$.

A number of other special classes of nonanalytic functions are determined in [25] and [26].

3.5. Functions which are analytic in the sense of operator K

A natural generalisation of analytic and c-analytic functions present functions which are analytic in the sense of operator K. They are the functions $w(z, \bar{z})=u(x, y)+iv(x, y)$ whose real and imaginary parts satisfy the following system

$$A(x, y)\frac{\partial u}{\partial x}-B(x, y)\frac{\partial v}{\partial y}=0, \quad A(x, y)\frac{\partial v}{\partial x}+B(x, y)\frac{\partial u}{\partial y}=0,$$

where A and B are given functions.

Putting $A=B$, we get the class of analytic functions, while for $A=-B$, we get the class of c-analytic functions.

Multiplying the second equation of the above system by i, and adding it to the first, we get

$$(A+B)\frac{\partial w}{\partial \bar{z}}+(A-B)\frac{\partial w}{\partial z}=0. \tag{3.16}$$

Since (3.16) is a linear homogenoeus partial diffeıential equation, it can always be reduced to an ordinary differential equation, so that we can then determine the classes of functions which are analytic in the sense of opeıator K.

For example, if A and B are constants, the corresponding functions which are analytic in the sense of operator K have the form

$$f\big((A-B)\bar{z}+(A+B)z\big).$$

3.6. Two properties of compound analytic functions

c-analytic functions, the simplest nonanalytic functions, are obtained from analytic functions by a formal substitution of $\bar{z}$ for z.

A much more general class of nonanalytic functions is obtained by a formal substitution of $h(z, \bar{z})$ for z, where h is a complex function.

Let $h(x, y)=C$ be the general solution of the differential equation $y'=\frac{g(x, y)}{f(x, y)}$. Then h satisfies the following equation

$$f(z, \bar{z})\frac{\partial w}{\partial z}+g(z, \bar{z})\frac{\partial w}{\partial \bar{z}}=0. \tag{3.17}$$

The general solution of (3.17) is

$$w=F\big(h(z, \bar{z})\big),$$

where F is an arbitrary differentiable function.

Complex function w given by (3.17) need not be an analytic function. Such functions we shall call *compound analytic functions.*

We shall give two theorems for such functions, which are analogous to the corresponding theorems which hold for analytic functions.

Theorem 3.3. *If w is a compound analytic function in a simply connected region R, and if C is a closed contour lying entirely inside R, then*

$$\int_C w(z, \bar{z})\, dh=0.$$

Proof. Let $f(z, \bar{z})=f_1(x, y)+if_2(x, y)$, and $g(z, \bar{z})=g_1(x, y)+ig_2(x, y)$. Equation (3.17) can be written in the form

$$(f_1+if_2)\big(u_x+v_y+i(v_x-u_y)\big)+(g_1+ig_2)\big(u_x-v_y+i(v_x+u_y)\big)=0,$$

and it separates into the system

$$(3.18)\qquad \begin{aligned} &(f_1+g_1)\,u_x+(f_2-g_2)\,u_y-(f_2+g_2)\,v_x+(f_1-g_1)\,v_y=0,\\ &(f_2+g_2)\,u_x-(f_1-g_1)\,u_y+(f_1+g_1)\,v_x+(f_2-g_2)\,v_y=0 \end{aligned}$$

Put

$$(3.19)\qquad a=f_1+g_1,\quad b=f_2-g_2,\quad c=f_2+g_2,\quad d=f_1-g_1.$$

Then

$$\begin{aligned}
\int_C w\,dh &= \int_C (u+iv)\left(\frac{\partial h}{\partial z}\,dz+\frac{\partial h}{\partial \bar z}\,d\bar z\right)\\
&=\frac{1}{2}\int_C (u+iv)\Bigg[\left(\frac{\partial h_1}{\partial x}+\frac{\partial h_2}{\partial y}+i\left(\frac{\partial h_2}{\partial x}-\frac{\partial h_1}{\partial y}\right)\right](dx+idy)\\
&\quad+\left[\frac{\partial h_1}{\partial x}-\frac{\partial h_2}{\partial y}+i\left(\frac{\partial h_2}{\partial x}+\frac{\partial h_1}{\partial y}\right)\right](dx-idy)\Bigg)\\
&=\int_C (u+iv)\left[\left(\frac{\partial h_1}{\partial x}\,dx+\frac{\partial h_1}{\partial y}\,dy\right)+i\left(\frac{\partial h_2}{\partial x}\,dx+\frac{\partial h_2}{\partial y}\,dy\right)\right]\\
&=\int_C u\frac{\partial h_1}{\partial x}\,dx+u\frac{\partial h_1}{\partial y}\,dy-v\frac{\partial h_2}{\partial x}\,dx-v\frac{\partial h_2}{y\partial}\,dy\\
&\quad+i\int_C v\frac{\partial h_1}{\partial x}\,dx+v\frac{\partial h_1}{\partial y}\,dy+u\frac{\partial h_2}{\partial x}\,dx+u\frac{\partial h_2}{\partial y}\,dy\\
&=\int_C \left(u\frac{\partial h_1}{\partial x}-v\frac{\partial h_2}{\partial x}\right)dx+\left(u\frac{\partial h_1}{\partial y}-v\frac{\partial h_2}{\partial y}\right)dy\\
&\quad+i\int_C \left(v\frac{\partial h_1}{\partial x}+u\frac{\partial h_2}{\partial x}\right)dx+\left(v\frac{\partial h_1}{\partial y}+u\frac{\partial h_2}{\partial y}\right)dy\\
&=\iint\left[\frac{\partial}{\partial x}\left(u\frac{\partial h_1}{\partial y}-v\frac{\partial h_2}{\partial y}\right)-\frac{\partial}{\partial y}\left(u\frac{\partial h_1}{\partial x}-v\frac{\partial h_2}{\partial x}\right)\right]dx\,dy\\
&\quad+i\iint\left[\frac{\partial}{\partial x}\left(v\frac{\partial h_1}{\partial y}+u\frac{\partial h_2}{\partial y}\right)-\frac{\partial}{\partial y}\left(v\frac{\partial h_1}{\partial x}+u\frac{\partial h_2}{\partial x}\right)\right]dx\,dy\\
&=\iint\left(\frac{\partial u}{\partial x}\frac{\partial h_1}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial h_2}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial h_1}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial h_2}{\partial x}\right)dx\,dy\\
&\quad+i\iint\left(\frac{\partial v}{\partial x}\frac{\partial h_1}{\partial y}+\frac{\partial u}{\partial x}\frac{\partial h_2}{\partial y}-\frac{\partial v}{\partial y}\frac{\partial h_1}{\partial x}-\frac{\partial u}{\partial y}\frac{\partial h_2}{\partial x}\right)dx\,dy\\
&=P_1+iP_2
\end{aligned}$$

where the double integrals are taken over the surface bounded by C. Let us determine the first one. Since w and h satisfy (3.17), i.e., since their real and imaginary parts satisfy the system (3.18), using notations (3.19) we get

$$P_1=\iint\Big(\frac{\partial u}{\partial x}\frac{\partial h_1}{\partial y}+\frac{(ab-cd)\,u_x+(b^2+d^2)\,u_y}{bc+ad}\,\frac{(a^2+c^2)\,h_{1x}+(ab-cd)\,h_1}{bc+ad}$$

$$-\frac{\partial u}{\partial y}\frac{\partial h_1}{\partial x}-\frac{(a^2+c^2)\,u_x+(ab-cd)\,u_y}{bc+ad}\,\frac{(ab-cd)\,h_{1x}+(b^2+d^2)\,h_1}{bc+ad}\Big)\,dx\,dy$$

$$=\iint\Big(\frac{\partial u}{\partial x}\frac{\partial h_1}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial h_1}{\partial x}+\frac{(b^2+d^2)\,(a^2+c^2)-(ab-cd)^2}{(bc+ad)^2}\,\frac{\partial u}{\partial y}\frac{\partial h_1}{\partial x}$$

$$+\frac{(ab-cd)^2\,(a^2+c^2)\,(b^2+d^2)}{(bc+ad)^2}\,\frac{\partial u}{\partial x}\frac{\partial h_1}{\partial y}\Big)\,dx\,dy.$$

Similarly, we can prove that $P_2=0$, which completes the proof of this theorem.

Theorem 3.4. *If we know the real part of a compound analytic function $w(z,\bar{z})=u(x,y)+iv(x,y)$, then we can determine its imaginary part up to a constant.*

Proof. Let u be known. Solving (3.18) with respect to v_x and v_y, and using (3.19), we get

$$v_x=\frac{(ab-cd)\,u_x+(b^2+d^2)\,u_y}{bc+ad},\qquad v_y=-\frac{(a^2+c^2)\,u_x+(ab-cd)\,u_y}{bc+ad}.$$

Now

$$dv=\frac{(ab-cd)\,u_x+(b^2+d^2)\,u_y}{bc+ad}\,dx-\frac{(a^2+c^2)\,u_x+(ab-cd)\,u_y}{bc+ad}\,dy.$$

This expression is a complete differential, which means that v can be determined up to a constant.

Remark. Notice that functions which are analytic in the sense of operator K are a special case of compound analytic functions.

3.7. Invariants of hyperbolic and elliptic partial differential equations

Using the partial derivatives $\frac{\partial}{\partial \bar{z}}$, $\frac{\partial}{\partial z}$ we can connect two important results of LAPLACE (published in 1777) and BURGATTI (published in 1895). See [27] and [28]. Namely, LAPLACE has proved that if one of the conditions

$$a_x+ab-c=0,\quad b_y+ab-c=0$$

is satisfied, then the hyperbolic equation

$$u_{xy}+au_x+bu_y+cu=0$$

can be integrated, i.e., one can find its general solution. BURGATTI has proved that if

$$\frac{1}{2}A_x+\frac{1}{2}B_y+\frac{A^2+B^2}{4}-C=0\ \text{ and }\ A_y-B_x=0,$$

then the equation

$$u_{xx}+u_{yy}+Au_x+Bu_y+C=0$$

can also be integrated. These two results follow, in fact, from each other. The proof of this can be found in [29].

4. SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

Definition 4.1. *We shall say that a system of k partial differential equations with k unknown functions is of type $k \times k$.*

4.1. Systems of type 2×2 — application of a real operator

The method considered in **2.** can be extended and applied to systems of partial differential equations. We shall illustrate the method by the systems of the form

$$f(x, y)\frac{\partial u}{\partial x}+g(x, y)\frac{\partial u}{\partial y}=a(\alpha)u+b(\alpha)v,$$

$$f(x, y)\frac{\partial v}{\partial x}+g(x, y)\frac{\partial v}{\partial y}=c(\alpha)u+d(\alpha)v,$$

where $\alpha(x, y)$ has the same meaning as in **1.1.**

In the system $(\mathcal{G}, A, X, \Phi)$ this system of equations reads

$$Au=au+bv, \qquad Av=cu+dv, \qquad (a, b, c, d \in \Phi) \tag{4.1}$$

where

$$A=f\frac{\partial}{\partial x}+g\frac{\partial}{\partial y}.$$

We shall look for the solution in the form

$$u=\lambda e^{rX}, \quad v=\mu e^{rX}.$$

Then

$$Au=\lambda r e^{rX}, \quad Av=\mu r e^{rX},$$

and substituting into (4.1) we get

$$(a-r)\lambda+b\mu=0, \qquad c\lambda+(d-r)\mu=0. \tag{4.2}$$

Algebraic system (4.2) will have nontrivial solutions in λ, μ only if

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix}=0,$$

from where we get two values for r (we suppose that $r_1 \neq r_2$). Putting each of those values into (4.2) we get λ_1, λ_2, μ_1, μ_2.

General solution of the system (4.1) is then

$$u=C_1\lambda_1 e^{r_1 X}+C_2\lambda_2 e^{r_2 X}, \qquad v=C_1\mu_1 e^{r_1 X}+C_2\mu_2 e^{r_2 X},$$

where C_1, C_2 are arbitrary elements of the set Φ.

For example, consider the system

$$y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=f_1(x^2-y^2)u+f_2(x^2-y^2)v,$$

$$y\frac{\partial v}{\partial x}+x\frac{\partial v}{\partial y}=f_3(x^2-y^2)u+f_4(x^2-y^2)v.$$

We have

$$X = \log(x+y), \qquad \Phi = \{f(x^2-y^2)\},$$

and, therefore, its general solution is given by

$$\begin{aligned} u(x, y) &= C_1(x^2-y^2)\,\lambda_1(x^2-y^2)\,(x+y)^{r_1(x^2-y^2)} \\ &\quad + C_2(x^2-y^2)\,\lambda_2(x^2-y^2)\,(x+y)^{r_2(x^2-y^2)}, \\ v(x, y) &= C_1(x^2-y^2)\,\mu_1(x^2-y^2)\,(x+y)^{r_1(x^2-y^2)} \\ &\quad + C_2(x^2-y^2)\,\mu_2(x^2-y^2)\,(x+y)^{r_2(x^2-y^2)}, \end{aligned}$$

where λ_i, μ_i, r_i $(i=1, 2)$ are determined in the same way as above (they are now functions of x^2-y^2), and C_1, C_2 are functions of the same argument.

4.2. Systems of type 2×2 — application of complex operators

We have already stated that Kolosov used his operator D for integration of systems of partial differential equations. So, for example, (see [12]) system

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=x, \quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=x-y. \tag{4.3}$$

after multiplying the second equation by i, and adding it to the first, Kolosov writes in the form

$$Dw=\frac{iz}{2}+\left(1+\frac{i}{2}\right)\bar{z}, \tag{4.4}$$

whereform, by analogy with ordinary differential equation

$$y' = C_1 + C_2 x,$$

whose solution is

$$y = C_1 x + C_2 \frac{x^2}{2} + C_3 ;$$

(C_3 is an arbitrary constant), he obtains a solution of (4.4) in the form of

$$w=\frac{iz\bar{z}}{4}+\frac{1}{4}\left(1+\frac{i}{2}\right)\bar{z}^2+\varphi(z) \tag{4.5}$$

where φ is an arbitrary analytic function. Separating the real and imaginary parts of (4.5) we finally get

$$\begin{aligned} u(x, y) &= \frac{1}{4}(x^2-y^2+xy)+\varphi_1(x, y) \\ v(x, y) &= \frac{3}{8}x^2+\frac{1}{8}y^2-\frac{1}{2}xy+\varphi_2(x, y), \end{aligned} \tag{4.6}$$

where φ_1, φ_2 are such that $\varphi_1+i\varphi_2$ is an analytic function.

Solution (4.6) is not the general solution of the system (4.3), as it does not contain two arbitrary functions. Namely, given the function φ_1, the other function φ_2 can be determined, up to a constant. Solution (4.6) depends, therefore, on an arbitrary function and an arbitrary constant.

Definition 4.2. *A solution of an n-th order system of partial differential equations*

$$F_1\left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \ldots, \frac{\partial^n v}{\partial y^n}\right)=0,$$

$$F_2\left(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \ldots, \frac{\partial^n v}{\partial y^n}\right)=0$$

will be called α-solution, if it contains n arbitrary functions and n arbitrary constants.

System (4.3) is naturally only an example for this method of integration. More generally, any system of the form

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=f(x, y, u, v), \qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=g(x, y, u, v), \tag{4.7}$$

where u and v are the unknown functions of x, y, and where f and g are given functions, under the condition

$$f(x, y, u, v)+ig(x, y, u, v)=F(z, \bar{z}, w), \tag{4.8}$$

reduces to an ordinary differential equation

$$y'=F(C, x, y), \tag{4.9}$$

where C is a parameter.

If

$$y=\Phi(x, C, C_1)$$

is the general solution of equation (4.9) (C_1 is the constant of integration), then the α-solution of (4.7) is given by

$$u(x, y)=\operatorname{Re}\Phi\left(\frac{\bar{z}}{2}, z, \varphi(z)\right), \qquad v(x, y)=\operatorname{Im}\Phi\left(\frac{\bar{z}}{2}, z, \varphi(z)\right),$$

where φ is an arbitrary analytic function.

Remark. System (4.7) can always be written in the form

$$Dw=f\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}, \frac{w+\bar{w}}{2}, \frac{w-\bar{w}}{2i}\right)+ig\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}, \frac{w+\bar{w}}{2}, \frac{w-\bar{w}}{2i}\right)$$
$$=F(z, \bar{z}, w, \bar{w}),$$

qut that equation is not an operator equation in the sense of Definition 0.2. Condition (4.8) is therefore necessary for this method of integration.

Various special cases of (4.7) are considered in papers [30], [31], [32]. For example, the following systems are solved:

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\operatorname{Re} f\left(z, \frac{w}{\bar{z}}\right), \quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\operatorname{Im} f\left(z, \frac{w}{\bar{z}}\right),$$

and

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\operatorname{Re}[f(z, \bar{z})\, g(z, w)], \quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\operatorname{Im}[f(z, \bar{z})\, g(z, w)],$$

etc., where the main idea was to reduce the integration of these systems to ordinary differential equations, whose solutions can be easily obtained.

Some more complicated systems were also solved, as for example (see [33]):

$$u=xP+yQ+\operatorname{Re}[f(P+iQ)], \quad v=xQ-yP+\operatorname{Im}[f(P+iQ)],$$

where f is a given function and

$$2P=u_x-v_y, \quad 2Q=v_x+u_y.$$

The above system reduces to CLAIRAUT's differential equation. In papers [34], [35], [26] some systems of second order were considered.

All these results are only technical realisations of KOLOSOV's idea.

Since the operator $\bar{D}$ is also a δ-operator, one can also obtain solutions of analogous systems of partial differential equations, i.e., of those systems, whose complex form is an operator equation involving $\bar{D}$.

Applying the operators K, $\bar{K}$ one can solve systems which are more general than those solvable by KOLOSOV's operators. Of course, we must always suppose that we know the solutions of the following two systems:

$$A\frac{\partial u}{\partial x}-B\frac{\partial v}{\partial y}=0, \quad A\frac{\partial v}{\partial x}+B\frac{\partial u}{\partial y}=0 \tag{4.10}$$

and

$$A\frac{\partial u}{\partial x}-B\frac{\partial v}{\partial y}=1, \quad A\frac{\partial v}{\partial x}+B\frac{\partial u}{\partial y}=0. \tag{4.11}$$

It was already shown (see 3.5) that (4.10) can always be written in the form

$$(A+B)\frac{\partial w}{\partial \bar{z}}+(A-B)\frac{\partial w}{\partial z}=0.$$

In the same way, (4.11) can be written as follows:

$$(A+B)\frac{\partial w}{\partial \bar{z}}+(A-B)\frac{\partial w}{\partial z}=\frac{1}{2}.$$

Since (4.10) and (4.11) are linear equations, their integration reduces to ordinary differential equations. In this way, integration od systems of partial differential equations, whose complex form is an operator equation in K, is reduced to integration of three ordinary differential equations.

For example, system

$$A\frac{\partial u}{\partial x}-B\frac{\partial v}{\partial y}=a(x, y)u-b(x, y)v+c(x, y),$$

$$A\frac{\partial v}{\partial x}+B\frac{\partial u}{\partial y}=b(x, y)u+a(x, y)v+d(x, y),$$

has the following complex form

$$Kw=Fw+G,$$

where $F(z, \bar{z})=a+ib,\ G(z, \bar{z})=c+id.$

As we have mentioned earlier, KOLOSOV's operator $D, \bar{D}$ satisfy the following equalities

$$D=2\frac{\partial}{\partial \bar{z}},\quad \bar{D}=2\frac{\partial}{\partial z}.$$

Up to now we have considered only those operator equations which involve D or $\bar{D}$ $\left(\text{i.e.}, \frac{\partial}{\partial \bar{z}}, \text{ or } \frac{\partial}{\partial z}\right)$. In fact, we can consider equations which contain both operators, or even more generally, which contain the expressions

$$\frac{\partial}{\partial z},\ \frac{\partial}{\partial \bar{z}},\ \frac{\partial^2}{\partial \bar{z}^2},\ \frac{\partial^2}{\partial z\,\partial \bar{z}},\ \frac{\partial^2}{\partial z^2},\ \ldots$$

In this way we obtain solutions of far more general systems of partial differential equations. In fact, we come to the following conclusion:

Every partial differential equation which contains a function in two variables corresponds to a system of two partial differential equations containing two unknown functions in two variables. If one knows the general solution of the partial differential equation, one also knows the α-solution of the corresponding systems, and vice versa.

We shall ill ustrate this by two examples.

EXAMPLE 4.2.1. System

$$a_{11}(x, y)\,u_x+a_{12}(x, y)\,u_y-b_{11}(x, y)\,v_x-b_{12}(x, y)v_y=f(x, y)\,u-g(x, y)\,v+h_1(x, y),$$

$$b_{11}(x, y)\,u_x+b_{12}(x, y)\,u_y+a_{11}(x, y)\,v_x+a_{12}(x, y)\,v_y=g(x, y)\,u+f(x, y)\,v+h_2(x, y)$$

can be reduced to the form

$$A(z, \bar{z})\frac{\partial w}{\partial \bar{z}}+B(z, \bar{z})\frac{\partial w}{\partial z}=C(z, \bar{z})\,w+D(z, \bar{z}), \tag{4.12}$$

where

$$A=\frac{a_{11}+b_{12}}{2}+i\frac{b_{11}-a_{12}}{2},\quad B=\frac{a_{11}-b_{12}}{2}+i\frac{b_{11}+a_{12}}{2},$$

$$C=\frac{1}{2}(f+ig),\quad D=\frac{1}{2}(h_1+ih_2).$$

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However, the following system of ordinary differential equations

$$\frac{d\bar{z}}{A}=\frac{dz}{B}=\frac{dw}{Cw+D} \tag{4.13}$$

corresponds to (4.12), and knowing the solution of (4.13) we can arrive at the solution of (4.12).

Let, for example, $A=-\bar{z}$, $B=2z$, $C=2$, $D=0$.

Then the general solution of (4.12) is

$$w=zf(\bar{z}z^2),$$

and therefore the α-solution of the corresponding system

$$\begin{aligned} x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}-3y\frac{\partial v}{\partial x}+3x\frac{\partial v}{\partial y}&=4u,\\ 3y\frac{\partial u}{\partial x}-3x\frac{\partial u}{\partial y}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}&=4v \end{aligned} \tag{4.14}$$

is given by

$$\begin{aligned} u(x, y)&=xf_1(x, y)-yf_2(x, y),\\ v(x, y)&=yf_1(x, y)+xf_2(x, y) \end{aligned} \tag{4.15}$$

where f_1+if_2 is an arbitrary compound analytic function, with $h(z, \bar{z})=\bar{z}z^2$. Therefore, according to Theorem 3.5, (4.15) presents the α-solution of (4.14).

4.3. Other types of systems

In using complex operators we have reduced systems of partial differential equations of first order to one partial differential equation of the form

$$F\left(z, \bar{z}, w, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial \bar{z}}\right)=0. \tag{4.16}$$

However, as we have mentioned earlier, we cannot reduce every system to (4.16). In fact, starting with an arbitrary first order system of partial differential equations, we can replace it by

$$F\left(z, \bar{z}, w, \bar{w}, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial \bar{z}}, \frac{\partial \bar{w}}{\partial z}, \frac{\partial \bar{w}}{\partial \bar{z}}\right)=0. \tag{4.17}$$

Equations of the form (4.17) cannot be integrated by analogy with real partial differential equations. Nevertheless, in some cases their solutions can be determined.

4.3.1. A system analogous to Položiĭ's p-system

POLOŽIĬ's p-sistem of partial differential equations reads

$$\frac{\partial u}{\partial x}=\frac{1}{p}\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{1}{p}\frac{\partial v}{\partial x},$$

where p is a given, positive and differentiable, function of x, y. Its complex form is

$$\frac{\partial w}{\partial \bar{z}} = P(z, \bar{z}) \frac{\partial \bar{w}}{\partial \bar{z}}$$

where

$$P(z, \bar{z}) = \frac{p-1}{p+1}.$$

$P(z, \bar{z})$ is clearly a real function. Suppose, however, that it is a complex, analytic function. We then get

$$\frac{\partial w}{\partial \bar{z}} = P(z) \frac{\partial \bar{w}}{\partial \bar{z}}. \tag{4.18}$$

We shall treat equation (4.18) as an undetermined partial differential equation.

Put $w = g(z, \bar{z}) = g_1 + ig_2$, where g is an arbitrary differentiable function. Equation (4.18) becomes

$$\frac{\partial w}{\partial \bar{z}} = P(z) \frac{\partial g}{\partial \bar{z}}$$

and this new equation can be integrated:

$$w = \int P(z) \frac{\partial g}{\partial \bar{z}} d\bar{z} = P(z) \int \frac{\partial g}{\partial \bar{z}} d\bar{z} = \alpha(z) + P(z) g(z, \bar{z}).$$

However, we have

$$\operatorname{Re} w = \operatorname{Re} \bar{w}, \qquad \operatorname{Im} w = -\operatorname{Im} \bar{w},$$

and therefore

$$\operatorname{Re}(\alpha + Pg) = \operatorname{Re} g, \qquad \operatorname{Im}(\alpha + Pg) = -\operatorname{Im} g,$$

i.e.,

$$\alpha_1 + p_1 g_1 - p_2 g_2 = g_1, \qquad \alpha_2 + p_1 g_2 + p_2 g_1 = -g_2.$$

If $p_1^2 + p_2^2 - 1 \neq 0$, this algebraic system always has solutions in g_1, g_2, and they are given by

$$g_1 = -\frac{\alpha_1 + \alpha_1 p_1 + \alpha_2 p_2}{p_1^2 + p_2^2 - 1}, \qquad g_2 = \frac{\alpha_2 - \alpha_2 p_1 + \alpha_1 p_2}{p_1^2 + p_2^2 - 1}.$$

Therefore, the general solution of (4.18) is given by $w = g_1 - ig_2$, and the α-solution of the system corresponding

$$(p_1 - 1) u_x + p_2 v_x - p_2 u_y + (p_1 + 1) v_y = 0,$$

$$p_2 u_x - (p_1 + 1) v_x + (p_1 - 1) u_y + p_2 v_y = 0,$$

by

$$u(x, y) = -\frac{\alpha_1 + \alpha_1 p_1 + \alpha_2 p_2}{p_1^2 + p_2^2 - 1}, \qquad v(x, y) = -\frac{\alpha_2 - \alpha_2 p_1 + \alpha_1 p_2}{p_1^2 + p_2^2 - 1}.$$

EXAMPLE 4.3.1.1. System

$$(4.19)\qquad \begin{aligned} &(x-1)u_x+yv_x-yu_y+(x+1)v_y=0,\\ &yu_x-(x+1)v_x+(x-1)u_y+yv_y=0 \end{aligned}$$

has the following complex form

$$\frac{\partial w}{\partial \bar z}=z\frac{\partial \bar w}{\partial \bar z}.$$

Putting $\bar w=g(z, \bar z)=g_1+ig_2$, we get

$$w=\alpha(z)+zg(z, \bar z),$$

where $\alpha=\alpha_1+i\alpha_2$ is an arbitrary analytic function.

However, $w=g_1-ig_2$, and therefore,

$$g_1=-\frac{\alpha_1+x\alpha_1+y\alpha_2}{x^2+y^2-1},\qquad g_2=\frac{\alpha_2-x\alpha_2+y\alpha_1}{x^2+y^2-1}$$

which means that the α-solution of (4.19) is given by

$$u(x, y)=g_1,\qquad v(x, y)=-g_2.$$

4.3.2. Položiĭ's p, q-system

As a generalisation of his p-system, POLOŽIĬ introduced (see [36], [37], [38]) the system

$$(4.20)\qquad pu_x+qu_y-v_y=0,\quad -qu_x+pu_y+v_x=0,$$

where p, q are given differentiable functions of x, y with $p>0$. System (4.20) defines the class of (p, q) — analytic functions.

Its complex form is

$$(4.21)\qquad (p+1-iq)\frac{\partial w}{\partial \bar z}+(p-1-iq)\frac{\partial \bar w}{\partial \bar z}=0.$$

Applying the same method as before, we see that it is possible to determine the α-solution of (4.20) if

$$\frac{p-1-iq}{p+1-iq}=P(z)$$

is an analytic function.

Indeed, putting

$$\bar w=g(z, \bar z)=g_1(x, y)+ig_2(x, y),$$

where g is a differentiable complex function, equation (4.21) becomes

$$\frac{\partial w}{\partial \bar z}+P(z)\frac{\partial g}{\partial \bar z}=0,$$

wherefrom we get

$$w=\alpha(z)-\int P(z)\frac{\partial g}{\partial \bar{z}}\,d\bar{z}=\alpha(z)-P(z)\int\frac{\partial g}{\partial \bar{z}}\,d\bar{z}=\alpha(z)-P(z)\,g(z,\bar{z}),$$

where $\alpha=\alpha_1+i\alpha_2$ is an arbitrary analytic function. However,

$$\operatorname{Re} w=g_1,\quad \operatorname{Im} w=-g_2,$$

i.e.,

$$\alpha_1-\frac{p^2+q^2-1}{(p^2+1)^2+q^2}g_1-\frac{2q}{(p^2+1)^2+q^2}g_2=g_1,$$

$$\alpha_2-\frac{p^2+q^2-1}{(p^2+1)^2+q^2}g_2+\frac{2q}{(p^2+1)^2+q^2}g_1=-g_2,$$

which gives the following algebraic system for g_1, g_2:

$$\frac{p^4+3p^2+2q^2}{(p^2+1)^2+q^2}g_1+\frac{2q}{(p^2+1)^2+q^2}g_2=\alpha_1,$$

$$-\frac{2q}{(p^2+1)^2+q^2}g_1-\frac{p^4+p^2+2}{(p^2+1)^2+q^2}g_2=\alpha_2.$$

Solving this system, we get

$$g_1=\frac{[(p^2+1)^2+q^2]\,[(p^4+p^2+2)\alpha_1+2q\alpha_2]}{(p^4+3p^2+2q^2)(p^4+p^2+2)-4q^2}$$

$$g_2=-\frac{[(p^2+1)^2+q^2]\,[(p^4+3p^2+2q^2)\alpha_2+2q\alpha_1]}{(p^4+3p^2+2q^2)(p^4+p^2+2)-4q^2}$$

which gives the α-solution of (4.20) in the form

$$u=g_1,\quad v=-g_2.$$

EXAMPLE 4.3.2.1. System

$$\text{(4.22)}\qquad \begin{aligned} &xu_x-yu_y-v_y=0,\\ &yu_x+xu_y+v_x=0,\end{aligned}$$

has the following complex form

$$\frac{\partial w}{\partial \bar{z}}+\frac{z-1}{z+1}\frac{\partial \bar{w}}{\partial \bar{z}}=0.$$

Since $(z-1)/(z+1)$ is an analytic function, we obtain the α-solution of (4.22) in the form

$$u(x,y)=\frac{[(x^2+1)^2+y^2][(x^4+x^2+2)\alpha_1-2y\alpha_2]}{(x^4+3x^2+2y^2)(x^4+x^2+2)-4y^2}$$

$$v(x,y)=\frac{[(x^2+1)^2+y^2]\,[(x^4+3x^2+2y^2)\alpha_2-2y\alpha_1]}{(x^4+3x^2+2y^2)(x^4+x^2+2)-4y^2}$$

where $\alpha_1(x,y)$ and $\alpha_2(x,y)$ are real and imaginary parts of an arbitrary analytic function.

4.3.3. Vekua's system

It is well known that VEKUA's system

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=au+bv+f,$$
(4.23)
$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=cu+dv+g,$$

can be written in the form

$$\frac{\partial w}{\partial \bar{z}}=Aw+B\bar{w}+C, \tag{4.24}$$

where

$$A=\frac{1}{4}[a+d+i(c-b)], \qquad B=\frac{1}{4}[a-d+i(c+b)], \qquad C=\frac{1}{2}(f+ig).$$

Case 1. $a=d,\ b+c=0$, i.e. $B=0$. The general solution of equation (4.24) is

$$w=e^{\int A(z,\bar{z})\,d\bar{z}}\Big(\alpha(z)+\int C(z,\bar{z})\,e^{-\int A(z,\bar{z})\,d\bar{z}}\Big)\,d\bar{z},$$

where α is an arbitrary analytic function. Therefore

$$u=\operatorname{Re} w, \qquad v=\operatorname{Im} w$$

presents the α-solution of system (4.23).

Case 2. Consider equation (4.24) together with

$$\frac{\partial \bar{w}}{\partial z}=\bar{A}\bar{w}+\bar{B}w+\bar{C}, \tag{4.25}$$

We shall look for the solution of equations (4.24) and (4.25), which we shall consider as a system with two unknown functions w and $\bar{w}$. Differentiating (4.24) with respect to z, we obtain

$$\frac{\partial^2 w}{\partial z\,\partial \bar{z}}=A_z w+Aw_z+B_z\bar{w}+B\bar{w}_z+C_z.$$

Elimination of $\bar{w}$ yields the LAPLACE equation

$$\frac{\partial^2 w}{\partial \bar{z}\,\partial z}-\frac{B_z+B\bar{A}}{B}\frac{\partial w}{\partial \bar{z}}-A\frac{\partial w}{\partial z}-\Big[A_z-\frac{A}{B}(B_z+B\bar{A})+B\bar{B}\Big]w \tag{4.26}$$
$$+\frac{C}{B}(B_z+B\bar{A})-B\bar{C}-C_z=0.$$

According to the general theory (see **[27]**), LAPLACE's equation

$$\frac{\partial^2 u}{\partial x\,\partial y}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+cu+d=0$$

is integrable if one of the following conditions is fulfilled

$$\frac{\partial b}{\partial y}+ab-c=0, \tag{4.27}$$

$$\frac{\partial a}{\partial x}+ab-c=0. \tag{4.28}$$

We shall apply these conditions to equation (4.26). Condition (4.27) becomes

$$-A_z+\frac{A}{B}(B_z+B\bar{A})+A_z-\frac{A}{B}(B_z+B\bar{A})+B\bar{B}=0,$$

i.e., $B\bar{B}=0$, or equivalently $B=0$. That is Case 1.

Condition (4.28) becomes

$$\frac{B_{\bar{z}}(B_z+B\bar{A})-B(B_{\bar{z}z}+B_{\bar{z}}A+B\bar{A}_z)}{B^2}+A_z+B\bar{B}=0$$

or, since we exclude $B=0$,

$$B_{\bar{z}}B_z-BB_{\bar{z}z}-B^2\bar{A}_z+B^2A_z+B^3\bar{B}=0.$$

In this case, we can obtain the general solution of (4.26), and then, using (4.24) we can find the corresponding value for $\bar{w}$.

Finally, putting

$$\operatorname{Re} w=\operatorname{Re}\bar{w}, \qquad \operatorname{Im} w=-\operatorname{Im}\bar{w},$$

we get the α-solution of (4.23) in the form of

$$u=\operatorname{Re} w, \qquad v=\operatorname{Im} w.$$

4.3.4. The connection between Vekua's and Položiĭ's p, q-system

Položiĭ's p, q-system

$$\begin{aligned} pu_x+qu_y-v_y&=0,\\ -qu_x+pu_y+v_x&=0, \end{aligned} \tag{4.29}$$

after the transformation (see [22])

$$U=pu, \qquad V=v-qu,$$

reduces to Vekua's system

$$\begin{aligned} U_x-V_y&=\frac{p_x+q_y}{p}U,\\ U_y+V_x&=\frac{p_y-q_x}{p}U \end{aligned} \tag{4.30}$$

whose complex form is

$$\frac{\partial w}{\partial \bar{z}} = \frac{p_x + q_y + i(p_y - q_x)}{4p} w + \frac{p_x + q_y + i(p_y - q_x)}{4p} \bar{w},$$

$$w = U + iV.$$

However, since the system (4.29) can be integrated if $\frac{p-1-iq}{p+1-iq}$ is an analytic function, the same holds for system (4.30).

Furthermore, system (4.30) can be integrated if

$$p_x + q_y + i(p_y - q_x) = 0,$$

i.e., if $q + ip$ is an analytic function, or if

$$B_{\bar{z}} B_z - B B_{\bar{z}z} - B^2 \bar{B}_z + B^2 B_z + B^3 \bar{B} = 0$$

where $B = \frac{p_x + q_y + i(p_y - q_x)}{4p}$, and in those cases system (4.29) can also be integrated.

4.3.5. Some more systems

Using similar procedures, we can solve systems whose complex form is, for example,

$$\frac{\partial w}{\partial z} = P(\bar{z}) \frac{\partial w}{\partial \bar{z}}$$

where $P(\bar{z})$ is a c-analytic function, or

$$\frac{\partial w}{\partial z} = Aw + B\bar{w} + C,$$

i.e., we can solve systems

$$(p_1 - 1) u_x + p_2 v_x + p_2 u_y - (p_1 + 1) v_y = 0,$$

$$p_2 u_x - (p_1 + 1) v_x - (p_1 - 1) u_y - p_2 u_y = 0,$$

where p_1, p_2 are given functions of x, y, where $p_1 + ip_2 = P(\bar{z})$, or, in the second case

$$u_x + v_y = au + bv + f, \qquad v_x - u_y = cu + dv + g$$

where

$$A = \frac{1}{4}[a + d + i(c - b)], \qquad B = \frac{1}{4}[a - d + i(c + b)], \qquad C = \frac{1}{2}(f + ig).$$

The analogy with the considered classes is clear, since it is only a question of a formal permutation of z and $\bar{z}$.

4.4. Hyperbolic systems

Up to now we have been considering only elliptic systems of partial differential equations. However, analogous results can be obtained for hyperbolic systems. The method which we shall expose here is more or less the same as the method described earlier.

We shall add to the set of real numbers an element, which we shall denote by the letter j. Let j satisfy all the operations of the set of real numbers. However, every appearance of the expression j^2 will be replaced by 1.

For a number of the form $z=x+jy$, we shall say that its first part is x, and the second part is y.

To a number $z=x+jy$, there corresponds a number $x-jy$, which we shall denote by $\bar{z}$.

Remark. Introduction of j does not lead to a new structure. It serves only to separate the first and the second part of the number $x+jy$. This only means that $x+jy$ is an ordered pair (x, y).

In the further text we shall consider functions which map R_j into itself. (R_j is the set of numbers of the form $x+jy$). Those are functions of the form

$$f(x+jy)=u(x, y)+jv(x, y).$$

We are especially interested in those functions which satisfy the system

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0, \tag{4.31}$$

We shall call them a-functions.

Let us prove the following theorem.

Theorem 4.1. *If the first part of an a-function is known, then the second part can be determined up to an arbitrary constant.*

Proof. Suppose that the first part of an a-function is known. Then, according to (4.31) we have

$$dv=-\frac{\partial u}{\partial y}dx-\frac{\partial u}{\partial x}dy.$$

This expression clearly represents a total differential, and hence

$$v=-\int\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy+C.$$

From this theorem we see that if we have a solution of a system of partial differential equations which contains two function $\alpha_1(x, y)$ and $\alpha_2(x, y)$ such that $\alpha_1+j\alpha_2$ is an arbitrary a-function, then that solution is an α-solution.

Let us now define for a function $w(z, \bar{z})=u(x, y)+jv(x, y)$ an operator which involves j:

$$D_j w=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+j\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right).$$

System $\left(R_j,\ D_j,\ \frac{z}{2},\ \{f(\bar{z})\}\right)$ is a δ-system.

We can now obtain α-solutions of hyperbolic systems in the same way as we have done with elliptic ones. We give an example which corresponds to KOLOSOV's example (see 4.2).

EXAMPLE. System

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=x, \qquad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=x-y, \tag{4.32}$$

after multiplying the second equation by j, and adding it to the first, becomes

$$D_j w = x+jx-jy=\bar{z}+\frac{j}{2}(z+\bar{z})=\left(1+\frac{j}{2}\right)\bar{z}+\frac{j}{2}z,$$

from where we get

$$w=\frac{1}{2}\left(1+\frac{j}{2}\right)\bar{z}z+\frac{j}{8}z^2+f(\bar{z})$$

where f is an arbitrary a-function. Separating the first and the second part of w, we get the α-solution of (4.32):

$$u(x, y)=\frac{1}{2}x^2-\frac{1}{2}y^2+\frac{1}{4}xy+\alpha(x, y),$$

$$v(x, y)=\frac{3}{8}x^2-\frac{1}{8}y^2+\beta(x, y).$$

4.5. Systems of type $2n\times 2n$

It is clear that all the methods which have been exposed here can be extended to systems of type $2n\times 2n$.

Namely, if we know the general solution of a system of ordinary or partial differential equations of type $n\times n$, we can then determine the α-solution of the corresponding system of partial differential equations of type $2n\times 2n$.

We shall only give one example to illustrate this method.

EXAMPLE. The general solution of the system

$$\frac{dx}{dt}=ax-y, \qquad \frac{dy}{dt}=x+ay$$

where a is a constant, is given by

$$x=e^{at}(c_1\sin t+c_2\cos t), \qquad y=e^{at}(c_2\sin t-c_1\cos t),$$

where c_1, c_2 are arbitrary constants.

However, system

$$\begin{aligned}
&\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=f_1(x, y)u-f_2(x, y)v-u_1,\\
&\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=f_2(x, y)u+f_1(x, y)v-v_1,\\
&\frac{\partial u_1}{\partial x}-\frac{\partial v_1}{\partial y}=u+f_1(x, y)u_1-f_2(x, y)v_1,\\
&\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y}=v+f_2(x, y)u+f_1(x, y)v,
\end{aligned} \tag{4.33}$$

can be written in the form

$$
(4.34)\qquad \begin{aligned} Dw &= f(z)\, w - w_1 \\ Dw_1 &= w + f(z)\, w_1, \end{aligned}
$$

where $w = u + iv$, $w_1 = u_1 + iv_1$, $f = f_1 + if_2$.

Suppose that f is an analytic function. Then the general solution of (4.34) is given by

$$
w = e^{f(z)\frac{\bar z}{2}} \left[\varphi_1(z) \sin \frac{\bar z}{2} + \varphi_2(z) \cos \frac{\bar z}{2} \right]
$$

$$
w_1 = e^{f(z)\frac{\bar z}{2}} \left[\varphi_2(z) \sin \frac{\bar z}{2} - \varphi_1(z) \cos \frac{\bar z}{2} \right],
$$

where $\varphi_1(z) = \alpha_1(x, y) + i\beta_1(x, y)$, $\varphi_2(z) = \alpha_2(x, y) + i\beta_2(x, y)$ are arbitrary analytic functions. Separating the real and imaginary parts of w and w_1, we obtain the α-solution of (4.31) in the form

$$
\begin{aligned}
u(x, y) &\\
= \exp\left(\frac{xf_1 + yf_2}{2}\right) &\left\{ \cos \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\alpha_1 \operatorname{ch} \frac{y}{2} - \beta_2 \operatorname{sh} \frac{y}{2} \right) + \cos \frac{x}{2} \left(\alpha_2 \operatorname{ch} \frac{y}{2} + \beta_1 \operatorname{sh} \frac{y}{2} \right) \right] \right. \\
&\left. - \sin \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\beta_1 \operatorname{ch} \frac{y}{2} + \alpha_2 \operatorname{sh} \frac{y}{2} \right) + \cos \frac{x}{2} \left(\beta_2 \operatorname{ch} \frac{y}{2} - \alpha_1 \operatorname{sh} \frac{y}{2} \right) \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
v(x, y) &\\
= \exp\left(\frac{xf_1 + yf_2}{2}\right) &\left\{ \cos \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\beta_1 \operatorname{ch} \frac{y}{2} + \alpha_2 \operatorname{sh} \frac{y}{2} \right) + \cos \frac{x}{2} \left(\beta_2 \operatorname{ch} \frac{y}{2} - \alpha_1 \operatorname{sh} \frac{y}{2} \right) \right] \right. \\
&\left. + \sin \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\alpha_1 \operatorname{ch} \frac{y}{2} - \beta_2 \operatorname{sh} \frac{y}{2} \right) + \cos \frac{x}{2} \left(\alpha_2 \operatorname{ch} \frac{y}{2} + \beta_1 \operatorname{sh} \frac{y}{2} \right) \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
u_1(x, y) &\\
= \exp\left(\frac{xf_1 + yf_2}{2}\right) &\left\{ \cos \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\alpha_2 \operatorname{ch} \frac{y}{2} + \beta_1 \operatorname{sh} \frac{y}{2} \right) - \cos \frac{x}{2} \left(\alpha_1 \operatorname{ch} \frac{y}{2} - \beta_2 \operatorname{sh} \frac{y}{2} \right) \right] \right. \\
&\left. - \sin \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\beta_2 \operatorname{ch} \frac{y}{2} - \alpha_1 \operatorname{sh} \frac{y}{2} \right) - \cos \frac{x}{2} \left(\beta_1 \operatorname{ch} \frac{y}{2} + \alpha_2 \operatorname{sh} \frac{y}{2} \right) \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
v_1(x, y) &\\
= \exp\left(\frac{xf_1 + yf_2}{2}\right) &\left\{ \cos \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\beta_2 \operatorname{ch} \frac{y}{2} - \alpha_1 \operatorname{sh} \frac{y}{2} \right) - \cos \frac{x}{2} \left(\beta_1 \operatorname{ch} \frac{y}{2} + \alpha_2 \operatorname{sh} \frac{y}{2} \right) \right] \right. \\
&\left. + \sin \frac{xf_2 - yf_1}{2} \left[\sin \frac{x}{2} \left(\alpha_2 \operatorname{ch} \frac{y}{2} + \beta_1 \operatorname{sh} \frac{y}{2} \right) - \cos \frac{x}{2} \left(\alpha_1 \operatorname{ch} \frac{y}{2} - \beta_2 \operatorname{sh} \frac{y}{2} \right) \right] \right\}.
\end{aligned}
$$

REFERENCES

1. J. HADAMARD: *Lectures on Cauchy's Problem in Linear Partial Differential Equations.* New York 1962. (First edition 1923).

2. J. HADAMARD: *Divers types de conditions définies et d'équations aux dérivées partielles.* Encyclopédie Française, troisième partie: La Mathématique, Paris 1937, p. 1. 82—1.

3. J. DRACH: *Intégration logique des équations différentielles ordinaires.* Int. Congress of Math. Cambridge 1912.

4. J. DRACH: *Intégration des équations aux dérivées partielles du second ordre par l'usage explicite des caractéristiques d'Ampere*. Bologna 1928.

5. G. HEILBRONN: *Intégration des équations aux dérivées partielles du second ordre par la méthode de Drach*. Mémorial des Scineces Mathématiques № **129**, Paris 1955.

6. G. HEILBRONN: *Intégration des équations ordinaires par la méthode de Drach*. ibid. №. **133**, Paris 1956.

7. W. F. AMES: *Nonlinear Partial Differential Equations in Engineering*. New York-London 1965.

8. Н. САЛТИКОВ: *Теорија парцијалних jeднaчинa gpyīoī pega*. Београд 1952.

9. J. D. KEČKIĆ: *Sur le problème de Cauchy pour une classe des équations paraboliques*. These Publications № **320** — № **328** (1970), 39—42.

10. Г. КОЛОСОВ: *Объ одномъ приложенiи теорiи функцiи комплекснаго перемѣннаго къ плоской задачѣ математической теорiи упругости*. Юрьев 1909.

11. G. KOLOSSOFF: *Über einige Eigenschaften des ebenen Problems der Elastizitätstheorie*. Z. Math. Phys. **62** (1914), 384—409.

12. Г. КОЛОСОВ: *О сопряженных дифференцiальныхъ уравненiяхъ съ частными производными съ приложенiемъ ихъ къ рѣшенiю математической физики*. Известия Э. И. (Ann Inst. Électrot. Petrograd) **11** (1914), 179—189.

13. D. S. MITRINOVIĆ and J. D. KEČKIĆ: *From the history of nonanalytic functions*, These Publications № **274** — № **301** (1969), 1—8.

14. D. S. MITRINOVIĆ and J. D. KEČKIĆ: *From the history of nonanalytic functions, II*. ibid. № **302** — № **319** (1970), 33—38.

15. В. И. СМИРНОВ: *Курс высшей математики*, т. 3, ч. 2. Москва 1969.

16. S. FEMPL: *Areolarni polinomi kao klasa neanalitičkih funkcija čiji su realni i imaginarni delovi poliharmonijske funkcije*. Mat. Vesnik **1 (16)** (1964), 29—38.

17. J. D. KEČKIĆ: *Analytic and c-analytic functions*. Publ. Inst. Math. (Beograd) **9 (23)** (1969), 189—198.

18. A. R. FORSYTH: *Lehrbuch des Differential-Gleichungen*. Braunschweing 1912.

19. L. BERS and A. GELBART: *On a class of functions defined by partial differential equations*. Trans. Amer. Math. Soc. **56** (1944), 67—93.

20. L. BERS: *Partial differential equations and generalised analytic functions*. Proc. Nat. Acad. Sci. **36** (1950), 130—136 and **37** (1951), 42—47.

21. Г. Н. ПОЛОЖИЙ: *Обобщение теории аналитических функций комплексного переменного*. Издательство Киевского Университета 1965.

22. И. Н. ВЕКУА: *Обобщенные аналитические функции*. Москва 1959.

23. Б. РИМАН: *Сочинения*. Москва 1948.

24. L. HÖRMANER: *An Introduction to Complex Analysis is Several Variables*. Toronto-New York-London 1966.

25. J. D. KEČKIĆ: *On some classes of nonanalytic functions*. These Publications № **274** — № **301** (1969), 83—86.

26. J. D. KEČKIĆ: *On some systems of partial differential equations and on some classes of nonanalytic functions*. ibid. № **247** — № **273** (1969), 61—66.

27. E. GOURSAT: *Leçons sur l'intégration des équations aux dérivées partielles du second ordre a deux variables indépendantes*, t. II. Paris 1926.

28. P. BURGATTI: *Sull'equazioni lineari alle derivate parziali del 2.° ordine (tipo ellittico), e sopra una classificazione dei sistemi di linee ortogonali che si possono tracciare sopra una superficie*. Ann. Mat. Pura Appl. **23** (1895), 225—267.

29. J. D. KEČKIĆ: *Invariants of partial differential equations of elliptic and hyperbolic type*. These Publications № **301** — № **319** (1970), 95—98.

30. S. FEMPL: *Reguläre Lösungen eines Systems partieller Differentialglechungen*. Publ. Inst. Math. (Beograd) **4 (18)** (1964), 115—120.

31. S. FEMPL: *Über einige Systems partieller Differentialgleichungen*. These Publications № **143** — № **155** (1965), 9—12.

32. J. Д. КЕЧКИЋ: *О jeдној класи парцијалних jeднaчинa*. Мат. Весник **6 (21)** (1969), 71—73.

33. С. Фемпл: *Об одной системе уравнений в часных производных, решение которой приводится к интегрованию уравнения вида Клеро.* Дифференциальные уравнения **1** (1965), 698—700.

34. S. Fempl: *Areoläre Exponentialfunktion als Lösung einer Klasse Differentialgleichungen.* Publ. Inst. Math. (Beograd) **8** (**22**) (1968), 138—142.

35. S. Fempl: *Über einer partieller Differentialgleichung in der nicht analytische Funktionen erscheinen.* ibid. **9** (**23**) (1969), 115—122.

36. Г. Н. Положий: *Интегрирование по сопряженным переменным.* Труды Третьего Всесоюзного математического сьеда. т. 1. Изд. АН СССР, Москва 1956, 95—96.

37. Г. Н. Положий: *К вопросу о (p, q)-аналитеческих функциях комплексного переменного и их применениях.* Revue de Math. pures et appl. **2** (1957), 331—361.

38. Г. Н. Положий: *О (p, q)-аналитических функциях комплесного переменного и некоторых их приложения.* Сборник: Исследования по современным проблемам теорий функции комплексного переменого. Москва 1960, 483—515.