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328. THE STEFFENSEN INEQUALITY*

P. S. Bullen

In this note a stronger form of Jensen's inequality is obtained and used to give a new proof of Steffensen's inequality. The method developed is then applied to obtain integral analogues of the Rado and Popoviciu inequalities.

1. In a recent very interesting paper, [**2**], Professor MITRINOVIĆ has discussed in some detail an elementary inequality due to STEFFENSEN. This inequality is known to imply a generalisation of JENSEN's inequality for continuous convex functions, [**2**]. It is perhaps of some interest to note that conversely STEFFENSEN's inequality results from this property of convex functions. In a small way this note may answer the hope of Professor MITRINOVIĆ that his review would initiate some new contributions.

All functions in this note will be real-valued functions defined on the bounded interval $[a, b]$. If $a_1, a_2, \ldots$ are real numbers we will write $A_n = \sum_{k=1}^{n} a_k$ and define $A_0 = 0$.

2. The following properties of concave functions are well-known, [**1**, p. 18].

Lemma 1. (*a*) *If f is a non-increasing function and $F = \int_a^x f$, then F is concave.*

(*b*) *If F is a concave function, $x_1 < x_2$, $y_1 < y_2$, $x_1 \leqq y_1$, $x_2 \leqq y_2$, then*

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} \geqq \frac{F(y_2) - F(y_1)}{y_2 - y_1}.$$

(*c*) *A continuous function F is concave if and only if for any n positive numbers $a_1, \ldots, a_n$ and any $x_1, \ldots, x_n$,*

$$F\left(\frac{1}{A_n} \sum_{k=1}^{n} a_k x_k\right) \geqq \frac{1}{A_n} \sum_{k=1}^{n} a_k F(x_k). \tag{1}$$

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The extension of JENSEN's inequality, (1), due to STEFFENSEN, allows the numbers $a_1, \ldots, a_n$ to be any real numbers provided that

$$(2) \qquad A_n \neq 0 \quad \text{and} \quad 0 \leqq \frac{A_k}{A_n} \leqq 1 \qquad (1 \leqq k \leqq n).$$

As the following lemma shows such conditions are natural ones for weights to satisfy.

Lemma 2. *If $x_1 \leqq \cdots \leqq x_n$, then*

$$(3) \qquad x_1 \leqq \frac{1}{A_n} \sum_{k=1}^{n} a_k x_k \leqq x_n$$

if and only if the real numbers $a_1, \ldots, a_n$ satisfy (2).

Proof. By ABEL's summation formula,

$$\frac{1}{A_n} \sum_{k=1}^{n} a_k x_k = x_n - \frac{1}{A_n} \sum_{k=1}^{n-1} A_k (x_{k+1} - x_k)$$

from which the sufficiency of conditions (2) for the validity of inequality (3) are immediate.

Taking $x_1 = \cdots = x_k = -1$, $x_{k+1} = \cdots = x_n = 0$ $(1 \leqq k \leqq n)$, gives the necessity of (2).

Theorem 3. *If F is a continuous concave function and $x_1 \leqq \cdots \leqq x_n$ and if $a_1, \ldots, a_n$ are real numbers satisfying* (2), *then inequality* (1) *holds.*

Proof. First note that it is sufficient to suppose $n \geqq 3$ since the other cases are covered by Lemma 1 (c).

(i) Suppose then that $n = 3$ and that $a_1 \geqq 0$, $a_3 \geqq 0$, $a_2 = -a_1 + a$, $a \geqq 0$. Note then that $A_3 = a + a_3$, and $\dfrac{a_1}{a + a_3} \leqq 1$.

If we write $\bar{x} = \dfrac{a x_2 + a_3 x_3}{a + a_3}$ then clearly $x_2 \leqq \bar{x} \leqq x_3$ and inequality (1) reduces to

$$F\left(\bar{x} - \frac{a_1}{a + a_3}(x_2 - x_1)\right) \geqq \frac{a F(x_2) + a_3 F(x_3)}{a + a_3} - \frac{a_1}{a + a_3}(F(x_2) - F(x_1)).$$

An application of Lemma 1 (c) with $n = 2$ to the right-hand side of this inequality shows that it is sufficient to prove that

$$F\left(\bar{x} - \frac{a_1}{a + a_3}(x_2 - x_1)\right) \geqq F(\bar{x}) - \frac{a_1}{a + a_3}(F(x_2) - F(x_1))$$

which is obvious if $x_2 = x_1$; if $x_2 > x_1$ this last inequality is equivalent to

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} \geqq \frac{F(\bar{x}) - F\left(\bar{x} - \frac{a_1}{a + a_3}(x_2 - x_1)\right)}{\bar{x} - \left(\bar{x} - \frac{a_1}{a + a_3}(x_2 - x_1)\right)}.$$

But, as we have remarked, $\bar{x} \geqq x_2$, and it is easily seen that

$$\bar{x} - \frac{a_1}{a+a_3}(x_2 - x_1) \geqq x_1;$$

hence this last inequality is a consequence of Lemma 1 (b). This completes the proof for the case $n=3$.

(ii) Now suppose that $n>3$ and that the result has been proved for all $k\,(3 \leqq k < n)$.

Let $a_k \geqq 0\,(1 \leqq k < p)$, $a_p < 0$; then, by hypothesis, $a_p = -A_{p-1} + a$, $a \geqq 0$. If we put

$$\bar{x} = \frac{a x_p + a_{p+1} x_{p+1} + \cdots + a_n x_n}{A_n}$$

then by Lema 2, $x_p \leqq \bar{x} \leqq x_n$; also write $\tilde{x} = \frac{1}{A_{p-1}} \sum_{k=1}^{p-1} a_k x_k$, then clearly

$$x_1 \leqq \tilde{x} \leqq x_{p-1}.$$

With these notations inequality (1) reduces to

$$F\left(\frac{A_{p-1}}{A_n}\tilde{x} - \frac{A_{p-1}}{A_n}x_p + \bar{x}\right) \geqq \frac{A_{p-1}}{A_n}\left(\frac{1}{A_{p-1}}\sum_{k=1}^{p-1} a_k F(x_k)\right) - \frac{A_{p-1}}{A_n}F(x_p)$$
$$+ \frac{a F(x_p) + a_{p+1} F(x_{p+1}) + \cdots + a_n F(x_n)}{A_n}.$$

Applying Lemma 1(c) with $n=p-1$ to the first term on the right-hand side of this inequality, and the induction hypothesis to the last term we see that it is sufficient to prove that

$$F\left(\frac{A_{p-1}}{A_n}\tilde{x} - \frac{A_{p-1}}{A_n}x_p + \bar{x}\right) \geqq \frac{A_{p-1}}{A_n}F(\tilde{x}) - \frac{A_{p-1}}{A_n}F(x_p) + F(\bar{x}).$$

But this last inequality follows from the case $n=3$ of the theorem. This remark completes the proof.

If in Lemma 1 (c) inequality (1) is strict unless $x_1 = \cdots = x_n$, F is said to be strictly concave. It is not difficult to see that if we assume F to be strictly concave in Theorem 3 then again (1) is strict unless $x_1 = \cdots = x_n$.

Theorem 3 is an important extension of JENSEN's inequality and it allows us to extend the classical inequalities between weighted means to means whose weight statisfy (2). Thus if $x_k > 0\,(1 \leqq k \leqq n)$ we write as usual

$$M_n^{[r]}(x;a) = \left(\frac{1}{A_n}\sum_{k=1}^{n} a_k x_k^r\right)^{1/r} \qquad (r \neq 0, \quad |r| < +\infty),$$

$$= \left(\prod_{k=1}^{n} x_k^{a_k}\right)^{1/A_n} \qquad (r=0),$$

$$= \max(x_1, \ldots, x_n) \qquad (r = +\infty),$$

$$= \min(x_1, \ldots, x_n) \qquad (r = -\infty).$$

Corollary 4. *If* $0 \leqq x_1 \leqq \cdots \leqq x_n$, *and* $a_1, \ldots, a_n$ *satisfy* (2), *then for* $r<s$

$$M_n^{[r]}(x; a) \leqq M_n^{[s]}(x; a),$$

with equality if and only if $x_1 = \cdots = x_n$.

This result, which includes (3) as a special case is proved using Theorem 3 just as the classical result can be proved using Lemma 1 (c) [3].

3. We now use Lemma 1 (a), and Theorem 3 to prove the STEFFENSEN inequality.

Theorem 5. *If* f *and* g *are two integrable functions,* f *non-increasing* $0 \leqq g \leqq 1$ *then*

$$\int_{b-\Gamma}^{b} f \leqq \int_a^b fg \leqq \int_a^{a+\Gamma} f \tag{4}$$

where $\Gamma = \int_a^b g$.

Proof. The idea of the proof is firstly to obtain (4) for g in a certain class of step-functions, then for g RIEMANN integrable, and finally for g integrable.

(i) Let $a = a_0 < a_1 < \ldots < a_n = b$ be a partition of $[a, b]$ and suppose that g is the step-function

$$g(x) = c_k, \qquad a_k \leqq x < a_{k+1}, \qquad k = 0, 1, \ldots, n-1.$$

This of course implies that $0 \leqq c_k \leqq 1$, $k = 0, 1, \ldots, n-1$.

Then the right-hand inequality in (4) reduces to

$$F(a_0) + \sum_{k=0}^{n-1} c_k \big(F(a_{k+1}) - F(a_k)\big) \leqq F\Big(a_0 + \sum_{k=0}^{n-1} c_k (a_{k+1} - a_k)\Big), \tag{5}$$

where $F(x) = \int_a^x f$.

But by Lemma 1 (a) F is concave and (5) then follows from Theorem 3.

A similar argument can be used for the left-hand inequality in (4). Thus completing the proof of Theorem 5 for g in this class of step-functions.

(ii) Now suppose that $g = \lim g_k$, g_k a step function of the type considered in (i) $(k = 1, 2, \ldots)$. If then $\Gamma_k = \int_a^b g_k$; gives

$$\int_{b-\Gamma_k}^{b} f \leqq \int_a^b f g_k \leqq \int_a^{a+\Gamma_k} f.$$

Hence, noting that $|f g_k| \leqq |f|$, (4) follows by letting $k \to +\infty$.

In particular this proves Theorem 5 for g in the class RIEMANN integrable functions.

(iii) The argument used in (ii) can now be used to extend the result to all integrable g and so complete the proof of Theorem 3.

4. This well known procedure can be used to extend other results from sums to integrals. In particular the RADO and POPOVICIU inequalities [3], have integral analogues that can be obtained in this manner.

The following inequalities are known (see: [3]); if $r \leqq 1 \leqq s$, then

$$(6) \qquad W_n \{M_n^{[s]}(c; w) - M_n^{[r]}(c; w)\} \geqq W_{n-1} \{M_{n-1}^{[s]}(c; w) - M_{n-1}^{[r]}(c; w)\};$$

if $r \leq 0 \leq s$, then

$$(7) \qquad \left(\frac{M_n^{[s]}(c; w)}{M_n^{[r]}(c; w)}\right)^{W_n} \geqq \left(\frac{M_{n-1}^{[s]}(c; w)}{M_{n-1}^{[r]}(c; w)}\right)^{W_{n-1}}.$$

They generalize the RADO and POPOVICIU inequalities respectively. In fact more is known; if the expressions on the left-hand sides of (6) and (7) are used to define functions on sets of integers, then these set functions are respectively super-additive and logarithmically super-additive [3].

For any bounded positive measurable function f defined on the interval $[0, x]$, let us write

$$M^{[r]}(x) = \left(\frac{1}{x}\int_0^x f^r\right)^{1/r}, \qquad r \neq 0, \ |r| < \infty,$$

$$= \exp\left(\frac{1}{x}\int_0^x \log f\right), \qquad r = 0.$$

Theorem 6. (a) *If* $r \leqq 1 \leqq s$ *and if* $\varrho(x) = x\{M^{[s]}(x) - M^{[r]}(x)\}$ *then* ϱ *is monotonically increasing, further the associated interval function is super-additive.*

(b) *If* $r \leqq 0 \leqq s$ *and if* $\pi(x) = \left(\frac{M^{[s]}(x)}{M^{[r]}(x)}\right)^x$ *then* π *is monotonically increasing; further the associated interval function is logarithmically super-additive.*

Proof. (a) Let $0 < x < y$ and suppose that $a_0 = 0 < a_1 < \cdots < a_m = x < \cdots < a_n = y$ is any partition of $[0, y]$ that contains the point x. Define ϱ as in section 3 then if we put $w_k = a_{k+1} - a_k$ $(k = 0, 1, \ldots, n-1)$ repeated application of (6) is equivalent to $\varrho(x) \leqq \varrho(y)$, wich proves the first part of (a) for this class of step functions. The proof of the first part of (a) is completed using the argument of Theorem 3.

The final part of (a) can be proved in a similar way using the remarks made above.

(b) The proof of (b) proceeds as for the proof of (a) but using inequality (8).

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