PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 320 — № 328 (1970)

# 327. SOME INEQUALITIES CONCERNING A TETRAHEDRON*

*Gojko Kalajdžić*

## Notations

Let $P$ be a point inside the tetrahedron $A_1A_2A_3A_4$ and let

| | |
|---|---|
| $H_1, H_2, H_3, H_4$ | be the heights of the tetrahedron which correspond to vertices $A_1, A_2, A_3, A_4$; |
| $R_1, R_2, R_3, R_4$ | be the distances from $P$ to $A_1, A_2, A_3, A_4$ respectively; |
| $r_1, r_2, r_3, r_4$ | be the distances from $P$ to the faces opposite to $A_1, A_2, A_3, A_4$; |
| $b_1\ b_2, b_3, b_4$ | be the areas of the faces of the tetrahedron which are opposite to $A_1, A_2, A_3, A_4$; |
| $V_1, V_2, V_3, V_4$ | be the volumes $\frac{1}{3}b_1 r_1$, $\frac{1}{3}b_2 r_2$, $\frac{1}{3}b_3 r_3$, $\frac{1}{3}b_4 r_4$ respectively; |
| $\varrho_1, \varrho_2, \varrho_3, \varrho_4$ | be the radii of the escribed spheres of the tetrahedron $A_1A_2A_3A_4$; |
| $V$ | be the volume of the tetrahedron $A_1A_2A_3A_4$. |

Other notations will be given in the text.

We shall prove a number of inequalities concerning the tetrahedron which we have not found in literature.

**Theorem 1.** *For a tetrahedron we have*

(1)
$$\frac{R_1}{H_1}+\frac{R_2}{H_2}+\frac{R_3}{H_3}+\frac{R_4}{H_4}\geqq 3.$$

***Proof.*** Since

$$R_i\geqq\frac{V-V_i}{V}H_i \qquad (i=1, 2, 3, 4),$$

we have

$$\sum_{i=1}^{4}\frac{R_i}{H_i}\geqq\sum_{i=1}^{4}\frac{V-V_i}{V}=3.$$

Equality holds in (1) if and only if the tetrahedron is regular and if $P$ is its centre.

*) Presented May 18, 1970 by D. S. Mitrinović and R. R. Janić.

**Theorem 2.** *Let $P$ be an arbitrary point inside the regular tetrahedron $A_1A_2A_3A_4$, $P_i$ $(i=1, 2, 3, 4)$ its projections on the corresponding sides of the tetrahedron, and $B_i$ points on $PP_i$ such that $PB_i=\lambda PP_i$ $(i=1, 2, 3, 4;\ \lambda>0)$.*

*Then, if $V$ and $V'$ denote the volumes of tetrahedra $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ we have*

$$V' \leqq \left(\frac{\lambda}{3}\right)^3 V. \tag{2}$$

***Proof.*** Let $H$ be the height of $A_1A_2A_3A_4$ and $\overrightarrow{PP_i}=\vec{r_i}$ $(i=1, 2, 3, 4)$. Then

$$H=r_1+r_2+r_3+r_4 \text{ and } V=\frac{\sqrt{3}}{8}H^3. \tag{3}$$

If $V''$ is the volume of the tetrahedron $P_1P_2P_3P_4$, then

$$V'=\lambda^3 V'', \tag{4}$$

since tetrahedrons $B_1B_2B_3B_4$ and $P_1P_2P_3P_4$ are homothetic with respect to homothety $(P, \lambda)$.

Since

$$\sin \sphericalangle (\vec{r_i}, \vec{r_k})=\frac{2\sqrt{2}}{3}, \quad \cos \sphericalangle (\vec{r_i}\times\vec{r_k}, \vec{r_j})=\frac{\sqrt{6}}{3} \quad (i, j, k=1, 2, 3, 4;\ i\neq j\neq k\neq i),$$

we have

$$V''=\frac{1}{6}\sum_{i=1}^{4}[\vec{r_i}, \vec{r_{i+1}}, \vec{r_{i+2}}]=\frac{2\sqrt{3}}{27}\sum_{i=1}^{4} r_i r_{i+1} r_{i+2} \qquad (r_{i+4}=r_i). \tag{5}$$

Suppose that $r_4=\max r_i$ $(i=1, 2, 3, 4)$ and $r_4=\text{const}$. Then, according to (3) we hawe that $r_1+r_2+r_3=\text{const.}$, and therefore the product $r_1 r_2 r_3$ is the greatest when $r_1=r_2=r_3=r$.

On the other hand

$$2(r_1 r_2+r_2 r_3+r_3 r_1)=(r_1+r_2+r_3)^2-(r_1^2+r_2^2+r_3^2)\leqq\frac{2}{3}(r_1+r_2+r_3)^2=\text{const.},$$

equality holding if and only if $r_1=r_2=r_3=r$.

Furthermore, since $r_4\geqq\frac{1}{4}H$, $r_4+3r=H$, we have $r\leqq\frac{1}{4}H$, and according to (5) we find

$$V''=\frac{2\sqrt{3}}{27}\left(r_4(r_1r_2+r_2r_3+r_3r_1)+r_1r_2r_3\right),$$

i. e.,

$$V''\leqq\frac{2\sqrt{3}}{27}r^2(3r_4+r)=\frac{2\sqrt{3}}{27}r^2(3H-8r),$$

or, owing to $r^2(3H-8r)\leqq\frac{1}{16}H^3$,

$$V''\leqq\frac{2\sqrt{3}}{27}\,\frac{H^3}{16}$$

whence, by (3) and (4) we get (2).

Equality holds in (2) if and only if $P$ is the centre of the tetrahedron.

**Theorem 3.** *Let points $A_{ik}(i, k=1. 2, 3, 4;\ i\neq k;\ A_{ik}=A_{ki})$ divide the edges $A_i A_k$ of the tetrahedron $A_1A_2A_3A_4$ in ratio $1:\lambda$, or $\lambda:1$ $(\lambda>0)$. If $V'$ is the volume of the polyhedron with vertices $A_{ik}$ $(i, k=1, 2, 3, 4;\ i\neq k;\ A_{ik}=A_{ki})$, then*

$$V'\leqq\left(1-\frac{4\lambda\sqrt{\lambda}}{(1+\lambda)^3}\right)V. \tag{6}$$

***Proof.*** Let

$$A_iA_k=a_{ik},\quad A_iA_{ik}=a_{k_i},\quad a_{k_i}:a_{i_k}=p_{ki}=\lambda\ \left(\text{or}\ \frac{1}{\lambda}\right), \tag{7}$$

$(i, k=1, 2, 3, 4;\ i\neq k;\ p_{ki}\,p_{ik}=1;\ k_i=1, \dots, 12)$. If $V_i$ denotes the volume of the tetrahedron $A_iA_{ij}A_{ik}A_{il}$ $(i, j, k, l=1, 2, 3, 4$ and mutually different), then clearly

$$\frac{V_i}{V}=\frac{a_{j_i}a_{k_i}a_{l_i}}{a_{ij}\,a_{ik}\,a_{il}}. \tag{8}$$

Taking into account (7) and (8), and using the arithmetic-geometric inequality, we have

$$\begin{aligned}V'&=V-(V_1+V_2+V_3+V_4)\\&=\left(1-\sum_{i=1}^{4}\frac{a_{j_i}a_{k_i}a_{l_i}}{a_{ij}\,a_{ik}\,a_{il}}\right)V\qquad (i, j, k, l=1, 2, 3, 4 \text{ and different})\\&=\left(1-\sum_{i=1}^{4}\frac{1}{(1+p_{ij})(1+p_{ik})(1+p_{il})}\right)V\\&\leqq\left(1-4\left(\prod_{i=1}^{4}(1+p_{ij})(1+p_{ik})(1+p_{il})\right)^{-\frac{1}{4}}\right)V,\end{aligned}$$

i.e.,

$$V'\leqq\left(1-\frac{4\lambda\sqrt{\lambda}}{(1+\lambda)^3}\right)V$$

since $a_{i_k}+a_{k_i}=a_{ik}$, $(1+p_{ik})(1+p_{ki})=\dfrac{(1+\lambda)^2}{\lambda}$ $(i, k=1, 2, 3, 4;\ i\neq k)$.

This proves inequality (6). Equality holds in (6) if and only if $\lambda=1$, i.e., if $A_{ik}$ are midpoints of edges $A_iA_k$.

**Theorem 4.** *For a tetrahedron we have*

$$R_1+R_2+R_3+R_4\geqq 2\sum_{1\leqq i<k}^{4}\sqrt{r_i r_k}. \tag{9}$$

***Proof.*** Let the plane determined by $P$ and the edge $A_iA_k$ of the tetrahedron meet its opposite edge in $A_{ik}$ $(i, k=1, 2, 3, 4;\ i\neq k;\ A_{ik}=A_{ki})$.

Let $A_4P$ meet the plane $A_1A_2A_3$ in $A_4'$; the line $A_iA_4'$ meets the corresponding side of the triangle $A_1A_2A_3$ in $A_i'$ $(i=1, 2, 3)$; furthermore, let the line which passes through $A_i$ and is parallel to $A_4P$ meet the plane $PA_jA_k$ $(i, j, k=1, 2, 3$ and are mutually different) in $A_i''$.

Then

$$(10)\qquad \frac{A_4 P}{A_i A_i''}=\frac{A_4 A_{jk}}{A_{jk} A_i},\quad \frac{A_4' P}{A_i A_i''}=\frac{A_4' A_i'}{A_i A_i'}\qquad (i, j, k=1, 2, 3 \text{ and different}),$$

If we add all the three equalities of the first set, and then all th equalities of the second set, and divide the obtained equalities we obtain

$$\frac{A_4 P}{PA_4'}=\frac{A_4 A_{23}}{A_{23} A_1}+\frac{A_4 A_{31}}{A_{31} A_2}+\frac{A_4 A_{12}}{A_{12} A_3}\qquad (A_{ik}=A_{ki}),$$

where it is taken into account that in the triangle $A_1 A_2 A_3$ holds

$$\frac{A_4' A_1'}{A_1 A_1'}+\frac{A_4' A_2'}{A_2 A_2'}+\frac{A_4' A_3'}{A_3 A_3'}=1.$$

Therefore, we hawe

$$R_i \geqq r_i\left(\frac{A_i A_{jl}}{A_{jl} A_k}+\frac{A_i A_{lk}}{A_{lk} A_j}+\frac{A_i A_{kj}}{A_{kj} A_l}\right)\qquad (i, j, k, l=1, 2, 3, 4 \text{ and are mutually different}),$$

i.e.,

$$R_1+R_2+R_3+R_4 \geqq \sum_{1\leqq i<k}^{4}\left(r_i \frac{A_i A_{jl}}{A_{jl} A_k}+r_k \frac{A_k A_{jl}}{A_{jl} A_i}\right) \geqq 2 \sum_{1\leqq i<k}^{4} \sqrt{r_i r_k}.$$

Equality holds in (9) if and only if the tetrahedron is regular and $P$ is its centre.

**Theorem 5.** *For a tetrahedron we have*

$$(11)\qquad \frac{H_1}{\varrho_1}+\frac{H_2}{\varrho_2}+\frac{H_3}{\varrho_3}+\frac{H_4}{\varrho_4} \geqq 8.$$

***Proof.*** Let $b_i$ $(i=1, 2, 3, 4)$ be the areas of the corresponding faces of the tetrahedron $A_1 A_2 A_3 A_4$. Then

$$3V=b_{i+1}\varrho_i+b_{i+2}\varrho_i+b_{i+3}\varrho_i-b_i\varrho_i \qquad (i=1, 2, 3, 4;\quad b_{i+4}=b_i),$$

i.e.,

$$(12)\qquad \varrho_i=\frac{3V}{b_{i+1}+b_{i+2}+b_{i+3}-b_i}.$$

On the orther hand

$$(13)\qquad H_i=\frac{3V}{b_i}\qquad (i=1, 2, 3, 4).$$

From (12) and (13) we get

$$\frac{H_1}{\varrho_1}+\frac{H_2}{\varrho_2}+\frac{H_3}{\varrho_3}+\frac{H_4}{\varrho_4}=\sum_{i=1}^{4}\frac{b_{i+1}+b_{i+2}+b_{i+3}-b_i}{b_i}$$

i.e.,

$$\sum_{i=1}^{4}\frac{H_i}{\varrho_i} \geqq 4\left(\frac{\prod_{i=1}^{4}(b_{i+1}+b_{i+2}+b_{i+3})}{b_1 b_2 b_3 b_4}\right)^{\frac{1}{4}}-4 \geqq 8,$$

where we have twice used the arithmetic-geometric mean inequality.

Equality holds in (11) if and only if $b_1=b_2=b_3=b_4$, i. e., if the tetrahedron is equifacial.

**Theorem 6.** *If* $\varrho$ *is the radius of the inscribed sphere of the tetrahedron* $A_1A_2A_3A_4$, *then*

$$\varrho_1+\varrho_2+\varrho_3+\varrho_4 \geqq 8\varrho. \tag{14}$$

***Proof.*** Since

$$\varrho=\frac{3V}{b_1+b_2+b_3+b_4}, \quad \varrho_i=\frac{3V}{b_{i+1}+b_{i+2}+b_{i+3}-b_i},$$

we have

$$\frac{2}{\varrho}=\frac{1}{\varrho_1}+\frac{1}{\varrho_2}+\frac{1}{\varrho_3}+\frac{1}{\varrho_4},$$

and, using the arithmetic-harmonic mean inequality, we obtain (14).

Equality holds in (14) if and only if the tetrahedron is equifacial.

**Theorem 7.** *For a tetrahedron we have*

$$\sum_{1\leqq i<k}^{4}\frac{1}{\varrho_i\varrho_k}\leqq 6\sum_{i=1}^{4}\frac{1}{H_i^2}. \tag{15}$$

***Proof.*** We have

$$\begin{aligned}\sum_{1\leqq i<k}^{4}\frac{1}{\varrho_i\varrho_k} &\leqq \frac{3}{2}\sum_{i=1}^{4}\frac{1}{\varrho_i^2}\\ &=\frac{3}{2}\sum_{i=1}^{4}\left(\frac{b_{i+1}+b_{i+2}+b_{i+3}-b_i}{3V}\right)^2\\ &=\frac{2(b_1^2+b_2^2+b_3^2+b_4^2)}{3V^2}\\ &=6\sum_{i=1}^{4}\frac{1}{H_i^2},\end{aligned}$$

which proves inequality (15).

Equality holds in (15) if and only if the tetrahedron is equifacial.

**Theorem 8.** *If $G$ is the centroid, $O$ the centre and $R$ the radius of the circumscribed sphere and $d=GO$, we have*

$$A_1G+A_2G+A_3G+A_4G\leqq 4(R^2-d^2)^{\frac{1}{2}}. \tag{16}$$

***Proof.*** STEINER's theorem reads: if $G$ is the centroid of the points $A_1, \dots, A_n$ and $O$ is arbitrary then

$$\sum_{i=1}^{n} A_iO^2=\sum_{i=1}^{n} A_iG^2+n\cdot OG^2.$$

In virtue of the above theorem, we have

$$A_1G^2+A_2G^2+A_3G^2+A_4G^2=OA_1^2+OA_2^2+OA_3^2+OA_4^2-4\,GO^2 \tag{17}$$
$$=4(R^2-d^2).$$

Using (17) and the arithmetic-quadratic mean inequality, we find

$$A_1G+A_2G+A_3G+A_4G\leqq 2(A_1G^2+A_2G^2+A_3G^2+A_4G^2)^{\frac{1}{2}}$$
$$\leqq 4(R^2-d^2)^{\frac{1}{2}}.$$

Equality holds in (16) if an only if the tetrahedron is equifacial.

**Remark.** A more general proposition also holds: if $G$ is the centroid of the polyhedron $A_1A_2\ldots A_n$, $R$ radius of the smallest circumscribed sphere, $d$ the distance from $G$ to the centre $O$ of that sphere, we have

$$\sum_{i=1}^{n} A_iG\leqq n(R^2-d^2)^{\frac{1}{2}}.$$

In this case identity (17) reads

$$\sum_{i=1}^{n} A_iG^2=\sum_{i=1}^{n} A_iO^2-n\cdot OG^2.$$

**Theorem 9.** *If $L$ is the sum of the edges and $R$ the radius of the circumscribed sphere of the tetrahedron $A_1A_2A_3A_4$, then*

$$L\leqq 4\sqrt{6}\,R. \tag{18}$$

***Proof.*** Let $G_i$ $(i=1, 2, 3, 4)$ be the centroids of the corresponding faces, $O$ the centre of the circumscribed sphere and $G$ the centroid of tetrahedron $A_1A_2A_3A_4$. Then, by STEINER's theorem we have

$$\sum_{k=1}^{3} A_iA_{i+k}^2=\sum_{k=1}^{3} G_iA_{i+k}^2+3A_iG_i^2, \qquad (i=1, 2, 3, 4;\ A_{i+4}=A_i) \tag{19}$$

wherefrom we obtain

$$\sum_{i=1}^{4} A_iG^2=\frac{1}{4}\Bigl(\sum_{1\leqq i<k}^{4} A_iA_k^2\Bigr), \tag{20}$$

since $A_iG=\frac{3}{4}A_iG_i$ $(i=1, 2, 3, 4)$.

On the other hand again by STEINER's theorem,

$$\sum_{i=1}^{4} OA_i^2=\sum_{i=1}^{4} A_iG^2+4\,GO^2,$$

i.e., owing to (20)

$$16R^2=\sum_{1\leqq i<k}^{4} A_iA_k^2+16\,GO^2. \tag{21}$$

Since

$$GO \geqq 0 \text{ and } \sum_{1 \leqq i<k}^{4} A_i A_k^2 \geqq \frac{1}{6} \cdot L^2,$$

from (21) follows (18).

Equality holds in (18) if and only if the tetrahedron is regular.

**Theorem 10.** *If $t_i$ $(i=1, 2, 3, 4)$ are medians of tetrahedron $A_1A_2A_3A_4$ and $R$ radius of the circumscribed sphere, then*

$$t_1+t_2+t_3+t_4 \leqq \frac{16}{3} R. \tag{22}$$

***Proof.*** If we replace $A_i G$ by $\frac{3}{4} A_i G_i$ $(i=1, 2, 3, 4)$ in (20) we get

$$\sum_{i=1}^{4} A_i G_i^2 = \frac{4}{9} \sum_{1 \leqq i<k}^{4} A_i A_k^2, \tag{23}$$

i. e., by (21)

$$16 R^2 = \frac{9}{4} \sum_{i=1}^{4} A_i G_i^2 + 16 GO^2$$

$$\geqq \frac{9}{16} (t_1+t_2+t_3+t_4)^2$$

which implies (22).

Equality holds in (22) if and only if the tetrahedron is equifacial.

**Theorem 11.** *If $R$ is the radius of the circumscribed sphere of the tetrahedron $A_1A_2A_3A_4$ and $A_i A_k = a_{ik}$ $(i, k=1, 2, 3, 4;\ i \neq k;\ a_{ik}=a_{ki})$, then*

$$V \leqq \frac{\sqrt{6}}{108} (a_{14}a_{23}+a_{24}a_{31}+a_{34}a_{12})^{\frac{3}{2}}, \tag{24}$$

$$V \leqq \frac{\sqrt{3}}{216} \left( \sum_{1 \leqq i<k}^{4} a_{ik}^2 \right)^{\frac{3}{2}}, \tag{25}$$

$$V \leqq \frac{8\sqrt{3}}{27} R^3. \tag{26}$$

***Proof.*** Let $Q$ be the area of the triangle whose sides are $a_{14}a_{23}$, $a_{24}a_{31}$, $a_{34}a_{12}$; then, by the CRELLE—von STAUDT formula, we have

$$V = \frac{Q}{6R}. \tag{27}$$

Since

$$12\sqrt{3}\, Q \leqq (a_{14}a_{23}+a_{24}a_{31}+a_{34}a_{12})^2,$$

and, according to (21)

$$4R \geqq \left( \sum_{1 \leqq i<k}^{4} a_{ik}^2 \right)^{\frac{1}{2}} \geqq 4 (a_{14}a_{23}+a_{24}a_{31}+a_{34}a_{12})^{\frac{3}{2}}, \tag{28}$$

using (27), we have

$$V \leqq \frac{\sqrt{6}}{108}\,(a_{14}a_{23}+a_{24}a_{31}+a_{34}a_{12})^{\frac{3}{2}}.$$

This proves inequality (24).

Since

$$2a_{ik}a_{jl} \leqq a_{ik}^2 + a_{jl}^2 \qquad (i, j, k, l = 1, 2, 3, 4 \text{ and different}),$$

(25) follows immediately from (24).

Inequality (26) follows from (25), by virtue of (28).

Equality holds in all inequalities (24), (25) and (26) if and only if the tetrahedron is regular.

*

* *

Professer O. Bottema has been kind enough to read the manuscript, has given a number of remarks and suggestions which have improved the text, for which the author is very thankful.

Katedra za matematiku
Elektrotehnički fakultet
Beograd, Jugoslavija